1. Suppose a spring is arranged horizontally. Let $y = y(t)$ be the position of the mass at the end of the spring relative to the equilibrium position. Suppose the mass is $m = 1$ kg, the damping constant is $\beta = 5$ and the spring constant is $k = 4$

a) *(5 points)*
Set up the differential equation that correctly determines $y(t)$.

$$y'' + 5y' + 4y = 0$$

b) *(20 points)* Use the method of Laplace transforms to solve your differential with the initial conditions $y(0) = 0$ and $y'(0) = 3$

$$\mathcal{L}\left(y''\right) + 5\mathcal{L}\left(y'\right) + 4\mathcal{L}\left(y\right) = \mathcal{L}\left(0\right)$$
$$s^2\mathcal{L}\left(y\right) - 3 + 5s\mathcal{L}\left(y\right) + 4\mathcal{L}\left(y\right) = 0$$
$$\mathcal{L}\left(y\right) = \frac{3}{(s+4)(s+1)} = \frac{1}{s+1} - \frac{1}{s+4} = \mathcal{L}\left(e^{-t}\right) - \mathcal{L}\left(e^{-4t}\right)$$
$$y = e^{-t} - e^{-4t}$$

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2. *(25 points)* Use the method of variation of parameters to find the general solution of the following differential equation. Show all work.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2$$

The homogeneous solutions are $y_{1h} = e^{0x} = 1$ and $y_{2h} = e^x$ so the Wronskian determinant is:

$$\mathcal{W} = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = e^x$$

Variation of parameters will produce two functions $v_1(x)$ and $v_2(x)$ so that:

$$y = v_1 y_{1h} + v_2 y_{2h} = v_1 \cdot 1 + v_2 e^x$$
$$\frac{dv_1}{dx} = e^{-x}\begin{vmatrix} 0 & e^x \\ 2 & e^x \end{vmatrix} = -2 \quad \text{so } v_1 = \int -2\,dx = -2x + C_1$$
$$\frac{dv_2}{dx} = e^{-x}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2e^{-x} \text{ so } v_2 = \int 2e^{-x}\,dx = -2e^{-x} + C_2$$
$$y = v_1 + v_2 e^x = -2x + C_1 + \left(-2e^{-x} + C_2\right)e^x = -2x + C_1 - 2 + C_2 e^x$$

If we let $a = C_1 - 2$ and $b = C_2$, we can simplify:

$$y = -2x + a + be^x$$

3. *(25 points)* Use the method of Laplace transforms to solve the following differential equation:

$$y'' - 8y' + 16y = 2e^{4t} \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 0$$

$$\mathcal{L}(y'') - 8\mathcal{L}(y') + 16\mathcal{L}(y) = \mathcal{L}\left(2e^{4t}\right)$$

$$s^2\mathcal{L}(y) - 8s\mathcal{L}(y) + 16\mathcal{L}(y) = \frac{2}{s-4}$$

$$(s-4)^2\mathcal{L}(y) = \frac{2}{s-4}$$

$$\mathcal{L}(y) = \frac{2}{(s-4)^3} = \mathcal{L}\left(t^2e^{4t}\right)$$

$$y = t^2e^{4t}$$

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4. *(25 points)* Solve the following differential equation. The symbol $\mathcal{U}$ stands for the unit step function.

$$y'' + y = 5e^{2t}\mathcal{U}(t-2\pi) \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 0$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = 5\mathcal{L}\left(e^{2t}\mathcal{U}(t-2\pi)\right)$$

$$s^2\mathcal{L}(y) + \mathcal{L}(y) = 5e^{-2\pi s}\mathcal{L}\left(e^{2(t+2\pi)}\right)$$

$$\left(s^2+1\right)\mathcal{L}(y) = 5e^{4\pi}e^{-2\pi s}\cdot\frac{1}{s-2}$$

$$\mathcal{L}(y) = e^{4\pi}e^{-2\pi s}\left(\frac{5}{(s-2)(s^2+1)}\right) = e^{4\pi}e^{-2\pi s}\left(\frac{1}{s-2} - \frac{s+2}{s^2+1}\right)$$

$$\mathcal{L}(y) = e^{4\pi}e^{-2\pi s}\mathcal{L}\left(e^{2t} - \cos t - 2\sin t\right)$$

Here is where we use the formula $\mathcal{L}(g(t-a)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t))$ where $a = 2\pi$ and $g(t) = e^{2t} - \cos t - 2\sin t$.

$g(t-2\pi) = e^{2(t-2\pi)} - \cos(t-2\pi) - 2\sin(t-2\pi) = e^{2t-4\pi} - \cos t - 2\sin t$.

$$\mathcal{L}(y) = e^{4\pi}\mathcal{L}(\mathcal{U}(t-2\pi)g(t-2\pi))$$

$$y = e^{4\pi}\mathcal{U}(t-2\pi)\left(e^{2t-4\pi} - \cos t - 2\sin t\right)$$

Simplify. The final answer is:

$$y = \mathcal{U}(t-2\pi)\left(e^{2t} - e^{4\pi}(\cos t + 2\sin t)\right)$$