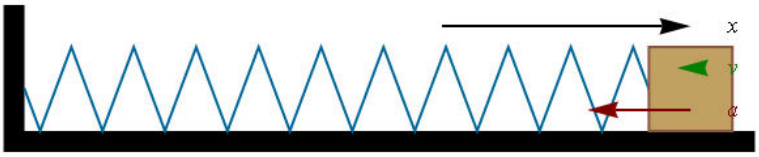
$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = \cos\gamma t$$

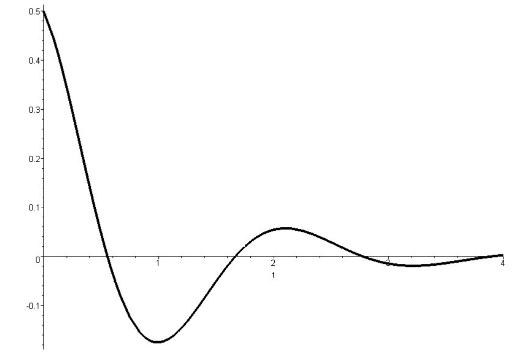


$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 9y = \cos 3t \qquad y(0) = \frac{1}{2} \quad y'(0) = 0$$

$$y = \frac{1}{2}e^{-t}\cos(2\sqrt{2}t) + \frac{1}{6}\sin(3t)$$

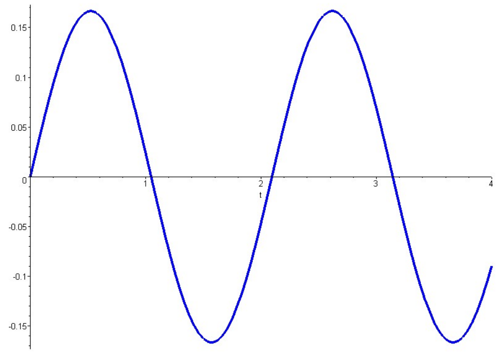
Homogeneous Solution

$$y_h = \frac{1}{2}e^{-t}\cos(2\sqrt{2}t)$$



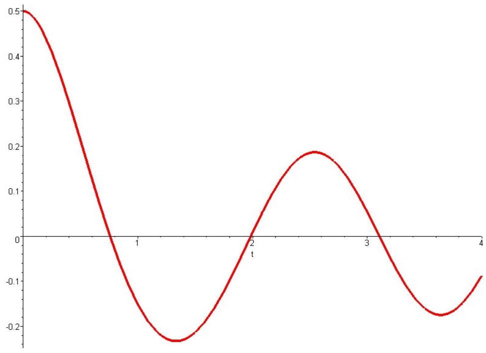
Particular Solution

$$y_p = \frac{1}{6}\sin(3t)$$



Homogeneous Plus Particular Solution

$$y = y_h + y_p$$



Now, let's solve the general case:

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = \cos\gamma t$$

$$y = y_h + y_p$$

$$y_h = ae^{-\frac{\beta t}{2m}}\cos\omega t + be^{-\frac{\beta t}{2m}}\sin\omega t$$

$$\text{where } \omega = \frac{\sqrt{4mk - \beta^2}}{2m}$$

To get the general form of y_p, use the annihilator of $\cos\gamma t$ to convert $\left(mD^2+\beta D+k\right)y=\cos\gamma t$ into a homogeneous equation:

$$\left(D^2+\gamma^2\right)\left(mD^2+\beta D+k\right)y=\left(D^2+\gamma^2\right)(\cos\gamma t)=0$$

Substitute $y=e^{rt}$

$$\left(r^2+\gamma^2\right)\left(mr^2+\beta r+k\right)=0$$

The solution of $r^2+\gamma^2=0$ will give us the particular solution.

$$y_p=c_1\cos\gamma t+c_2\sin\gamma t$$

Substitute $y_p = c_1 \cos \gamma t + c_2 \sin \gamma t$ into $my'' + \beta y' + ky = \cos \gamma t$ to determine the correct values of c_1 and c_2.

$$y_p = c_1 \cos \gamma t + c_2 \sin \gamma t$$

$$y_p' = c_2 \gamma \cos \gamma t - c_1 \gamma \sin \gamma t$$

$$y_p'' = -c_1 \gamma^2 \cos \gamma t - c_2 \gamma^2 \sin \gamma t$$

Substitute $y_p = c_1 \cos \gamma t + c_2 \sin \gamma t$ into $my'' + \beta y' + ky = \cos \gamma t$ to determine the correct values of c_1 and c_2.

$$ky_p = c_1 k \cos \gamma t + c_2 k \sin \gamma t$$

$$\beta y_p' = c_2 \beta \gamma \cos \gamma t - c_1 \beta \gamma \sin \gamma t$$

$$my_p'' = -c_1 m \gamma^2 \cos \gamma t - c_2 m \gamma^2 \sin \gamma t$$

Substitute $y_p = c_1 \cos\gamma t + c_2 \sin\gamma t$ into $my'' + \beta y' + ky = \cos\gamma t$ to determine the correct values of c_1 and c_2.

$$ky_p = c_1 k \cos\gamma t + c_2 k \sin\gamma t$$

$$\beta y_p' = c_2 \beta\gamma \cos\gamma t - c_1 \beta\gamma \sin\gamma t$$

$$my_p'' = -c_1 m\gamma^2 \cos\gamma t - c_2 m\gamma^2 \sin\gamma t$$

$$\begin{aligned} my_p'' + \beta y_p + ky =& (c_1(k - m\gamma^2) + c_2\beta\gamma) \cos\gamma t \\ &+ \left(-\beta\gamma c_1 + c_2 \left(k - m\gamma^2\right)\right) \sin\gamma t \end{aligned}$$

Substitute $y_p = c_1 \cos \gamma t + c_2 \sin \gamma t$ into $my'' + \beta y' + ky = \cos \gamma t$ to determine the correct values of c_1 and c_2.

$$ky_p = c_1 k \cos \gamma t + c_2 k \sin \gamma t$$

$$\beta y_p' = c_2 \beta \gamma \cos \gamma t - c_1 \beta \gamma \sin \gamma t$$

$$my_p'' = -c_1 m \gamma^2 \cos \gamma t - c_2 m \gamma^2 \sin \gamma t$$

$$\begin{aligned}\cos \gamma t =& (c_1 (k - m\gamma^2) + c_2 \beta \gamma) \cos \gamma t \\ &+ \left(-\beta \gamma c_1 + c_2 \left(k - m\gamma^2\right)\right) \sin \gamma t\end{aligned}$$

Solve for c_1 and c_2

$$\begin{aligned} \left(k - m\gamma^2\right) c_1 \qquad + \qquad \beta\gamma c_2 &= 1 \\ -\beta\gamma c_1 + \left(k - m\gamma^2\right) c_2 &= 0 \end{aligned}$$

$$\begin{aligned} \left(k - m\gamma^2\right) c_1 \qquad + \qquad \beta\gamma c_2 &= 1 \\ -\beta\gamma c_1 + \left(k - m\gamma^2\right) c_2 &= 0 \end{aligned}$$

$$c_1 = \frac{k - m\gamma^2}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2} \qquad c_2 = \frac{\beta\gamma}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$\begin{aligned} \left(k - m\gamma^2\right) c_1 \quad &+ \quad \beta\gamma c_2 = 1 \\ -\beta\gamma c_1 + \left(k - m\gamma^2\right) c_2 &= 0 \end{aligned}$$

$$c_1 = \frac{k - m\gamma^2}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2} \qquad c_2 = \frac{\beta\gamma}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$y = ae^{-\frac{\beta t}{2m}} \cos \omega t + be^{-\frac{\beta t}{2m}} \sin \omega t + c_1 \cos \gamma t + c_2 \sin \gamma t$$

Please note that this solution assumes $\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2 \neq 0$