Differential Equations and Matrix Methods

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Today's Topic : Determinants

$$ax + by = f_1$$
$$cx + dy = f_2$$

Solution:

$$x = \frac{f_1 d - f_2 b}{ad - bc} \qquad y = \frac{f_2 a - f_1 c}{ad - bc}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Alternative notation. If $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then:

$$\det(\mathbf{A}) = ad - bc$$

$$\begin{aligned} ax + by &= f_1 \\ cx + dy &= f_2 \end{aligned}$$

Solution:

$$x = \frac{f_1 d - f_2 b}{ad - bc} = \frac{\begin{vmatrix} f_1 & b \\ f_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$ax + by = f_1$$
$$cx + dy = f_2$$

Solution:

$$y = \frac{f_2 a - f_1 c}{ad - bc} = \frac{\begin{vmatrix} a & f_1 \\ c & f_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let M_{ij} denote the determinant obtained if the i^{th} row and j^{th} column is deleted. This is called the Minor of the i^{th} row, j^{th} column.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$M_{11} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$M_{11} = a_{22}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$M_{12} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$M_{12} = a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$
$$= a_{11}M_{11} - a_{12}M_{12}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{11} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\text{Calculate} \quad \begin{vmatrix} 4 & 3 & 2 \\ 1 & 3 & 0 \\ 2 & 2 & 1 \end{vmatrix}$$

$$\text{Calculate} \quad \begin{vmatrix} 4 & 3 & 2 \\ 1 & 3 & 0 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}$$

$$= 4(3-0) - 3(1-0) + 2(2-6) = 12 - 3 - 8 = 1$$

$$\text{Calculate} \quad \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= 0\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}$$

$$= 0(3-0) - 0(1-0) + 0(2-6) = 0$$

$$\vec{\mathbf{u}} = 1\vec{\mathbf{i}} + 2\vec{\mathbf{j}} + 1\vec{\mathbf{k}} \qquad \vec{\mathbf{v}} = 0\vec{\mathbf{i}} - 1\vec{\mathbf{j}} + 2\vec{\mathbf{k}}$$

Calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$

$$\begin{aligned}
\vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \vec{\mathbf{k}} \\
&= 5\vec{\mathbf{i}} - 2\vec{\mathbf{j}} - \vec{\mathbf{k}}
\end{aligned}$$

Algorithm for finding the inverse of a matrix

$$(\mathbf{A} \mid \mathbf{I}) \xrightarrow{\textbf{reduce}} (\mathbf{I} \mid \mathbf{A}^{-1})$$

If **A** has an inverse, it can be reduced to the identity matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

How do the elementary row operations effect the value of a determinant?

1. Interchange two rows
2. Multiply a row by a nonzero constant
3. Add a multiple of one row to another.

Interchange two rows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{12}a_{21} - a_{11}a_{22}$$

Interchange two rows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{12}a_{21} - a_{11}a_{22}$$

Interchanging two rows has the effect of changing the determinant by a minus sign

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix} = ka_{11}a_{22} - ka_{12}a_{21}$$

Multiplying a row by a non-zero constant multiplies the entire determinant by that constant.

$$\begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Add a multiple of one row to another

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} + ka_{11} & a_{22} + ka_{12} \end{vmatrix}$$

Add a multiple of one row to another

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} + ka_{11} & a_{22} + ka_{12} \end{vmatrix}$$

$$= a_{11}(a_{22}+ka_{12})-a_{12}(a_{21}+ka_{11}) = a_{11}a_{22}-a_{12}a_{21}$$

Adding a multiple of one row to another does not change the value of the determinant

Interchanging two rows has the effect of changing the determinant by a minus sign

Multiplying a row by a non-zero constant multiplies the entire determinant by that constant.

Adding a multiple of one row to another does not change the value of the determinant

$$\mathbf{A} \longrightarrow \mathbf{B}$$

Suppose matrix **A** is transformed into matrix **B** by interchanging two rows

$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

If det(**A**)$=0$ then det(**B**) will be 0 also.
If det(**A**) is not 0 then det(**B**) won't be 0 either

$$\mathbf{A} \longrightarrow \mathbf{B}$$

Suppose matrix $\mathbf{A}$ is transformed into matrix $\mathbf{B}$ by multiplying a row by a nonzero constant c

$$\det(\mathbf{B}) = c \cdot \det(\mathbf{A})$$

If $\det(\mathbf{A})= 0$ then $\det(\mathbf{B})$ will be 0 also.
If $\det(\mathbf{A})$ is not 0 then $\det(\mathbf{B})$ won't be 0 either

$$\mathbf{A} \longrightarrow \mathbf{B}$$

Suppose matrix $\mathbf{A}$ is transformed into matrix $\mathbf{B}$ by adding a multiple of one row to another row

$$\det(\mathbf{B}) = \det(\mathbf{A})$$

If $\det(\mathbf{A}) = 0$ then $\det(\mathbf{B})$ will be 0 also.
If $\det(\mathbf{A})$ is not 0 then $\det(\mathbf{B})$ won't be 0 either

$$\mathbf{A} \longrightarrow \mathbf{B}$$

Suppose matrix **A** is transformed into matrix **B** by performing a sequence of elementary row operations

If det(**A**)= 0 then det(**B**) will be 0 also.
If det(**A**) is not 0 then det(**B**) won't be 0 either

If $\mathbf{A}^{-1}$ exists then $\mathbf{A}$ can be reduced to the identity matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This can't happen if the determinant of $\mathbf{A}$ equals 0

Theorem: If det($\mathbf{A}$)$=0$ then $\mathbf{A}^{-1}$ doesn't exist

Example: Can we find the inverse of the following matrix?

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 3 & 3 \\ 1 & 5 & 4 \end{pmatrix}$$

$$\begin{aligned}\begin{vmatrix} 1 & 3 & 2 \\ 0 & 3 & 3 \\ 1 & 5 & 4 \end{vmatrix} &= 1\begin{vmatrix} 3 & 3 \\ 5 & 4 \end{vmatrix} - 3\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} + 2\begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} \\ &= 1(12-15) - 3(0-3) + 2(0-3) \\ &= 0\end{aligned}$$

The matrix has no inverse