MA 345 Differential Equations - Assignments **Dr. E. Jacobs**

Assignment 1. *Introduction to Differential Equations. Separation of Variables*

Read 1.1, 1.2, 2.1 and 2.2
You should be able to do the following problems:
Exercise 1.2 Problems 1 - 15, Exercise 2.2 Problems 7 - 26
Hand in the following problems:

Solve the following differential equations for problems 1 - 3.

1. $$\frac{dy}{dx} = \frac{y}{2\sqrt{x}} \qquad y(1) = 1$$

2. $$\frac{dy}{dx} = \frac{y(y+1)}{x}$$

3. $$\frac{dy}{dx} + xy = x \qquad y(0) = 2$$

4. Scientists at the University of Nebraska Medical Center performed an experiment to determine the rate at which pancreatic cancer cells grow. Approximately 500,000 pancreatic cancer cells were injected into the pancreas of each laboratory rat and the number of cells was observed growing over a three week period. If $y(t)$ represents the population of cancer cells in a particular rat after t hours, then $y(t)$ solves the following differential equation:

$$\frac{dy}{dt} = ky \qquad \text{(where } k \text{ is a positive constant)}$$

a. Solve this differential equation and obtain a formula for $y(t)$.

b. The scientists determined experimentally that $k = 0.05 \ln 2$. Calculate how long it takes for the population of cancer cells to double.

5. Let $M(t)$ denote the mass of a radioactive object after t years. The fact that the rate at which the mass is decreasing is proportional to the mass itself leads to the differential equation:

$$\frac{dM}{dt} = -\lambda M \quad \text{where } \lambda \text{ is a constant}$$

a. Solve this differential equation and obtain a formula for $M(t)$.

b. Suppose for a particular radioactive substance we begin with 4 grams. If the mass is down to 1 gram after 1 year, calculate the constant λ

Assignment 2. *Exact Differential Equations*

Read 2.4
You should be able to do the following problems:
Exercise 2.4 Problems 9 - 26
Hand in the following problems:

Find the solution of each of the following equations. In each case, it is possible to solve for y in terms of x, so you should do it to complete the problem.

1.
$$(e^x + 2xe^y)\,dx + x^2 e^y\,dy = 0$$

2.
$$(\sin y - \sin x)\,dx + (1+x)\cos y\,dy = 0 \qquad y(0) = 0$$

3.
$$\left(\sqrt{y} - 2x\right)\,dx + \left(\frac{x+1}{2\sqrt{y}}\right)\,dy = 0 \qquad y(0) = 1$$

4.
$$(2x + \ln y)dx + \frac{x}{y}dy = 0 \qquad y(1) = 1$$

5.
$$\frac{dy}{dx} = \frac{1 - 2x + 2y}{2y - 2x}$$

Assignment 3. *Integrating Factors and Special Substitutions*

Read 2.5 and 2.6

You should be able to do the following problems:

Exercise 2.5 Problems 7 - 12

Hand in the following problems:

Find the solutions of each of the following equations.

1. $$x^2 \frac{dy}{dx} = y^2 + xy \qquad \textit{Hint: Try } v = \frac{y}{x}$$

2. $$(e^x + y)\, dx + x\, dy = 0$$

3. $$\left(e^{2x} - y^3\right)\, dx + 3y^2\, dy = 0 \qquad y(0) = 0$$

4. $$(2xy - 1)\, dx + \frac{x}{y}\, dy = 0$$

5. $$\left(\frac{1}{y} + x\right)\, dx - \frac{1}{y^2}\, dy = 0$$

Assignment 4. *First Order Linear Differential Equations*

Read 2.3

You should be able to do the following problems:

Exercise 2.3 Problems 7 - 25

Hand in the following problems:

Find the solutions of each of the following equations

1. $$\frac{dy}{dx} - 4y = e^{3x} \qquad y(0) = 0$$

2. $$\frac{dy}{dx} = x + y \qquad y(0) = 2$$

3. $$x\frac{dy}{dx} = \tan x - y$$

4. $$\left(x + x^2\right)\frac{dy}{dx} + xy = 1 + x$$

5. $$\frac{dy}{dx} - \frac{2y}{x} = 1$$

Assignment 5. *Applications of First Order Differential Equations*

Read 3.1 and 3.2

You should be able to do the following problems:

Exercise 3.2 Problems 1 - 25

Hand in the following problems:

1. A small object is brought into a room. The room temperature is 20 degrees Centigrade. At $t = 0$, the temperature of the object is 10 degrees Centigrade. After $t = 1$ minute, the temperature of the object is 12 degrees. Let the temperature of the object after t minutes be denoted by $y(t)$. Assume the temperature of the room does not change significantly. The temperature of the object changes according to the following law:

The rate at which the temperature rises is proportional to the temperature difference between the object and the environment

Write a differential equation that expresses how $y(t)$ changes. Solve the differential equation for $y(t)$. Determine all constants that appear in your solution.

2. The forces on a falling body add up to (mass)(acceleration)

$$m\frac{dv}{dt} = \text{(force of gravity)} + \text{(force due to air resistance)}$$

We can solve this differential equation and obtain a formula for the velocity $v = v(t)$. The force of gravity $= mg$. A somewhat oversimplified model for the force due to air resistance is that it equals $-\beta v$ where β is a constant. Assume that the falling object starts with an initial velocity of 0 and solve for $v(t)$. Your answer will be in terms of the constants m, g and β.

3. A 100 kilogram sailboat is floating motionless in the water. Suddenly, a wind with a constant force of 50 newtons begins to push the boat forward. The force of resistance of the water is proportional to the boats velocity, with a proportionality constant of $k = 25$.

Let $v(t)$ be the velocity of the sailboat after t seconds.

Write an appropriate differential equation that determines $v(t)$ and solve it.

4. A basin holds 4 liters of water at $t = 0$. Then, we start adding brine through an inflow pipe at the rate of 1 liter/minute. The brine coming through the inflow pipe contains salt at a concentration of $\frac{1}{4}$ gram/liter. At the same time, brine is flowing out of the basin through an outflow pipe at the rate of 1 liter/minute.

Let $x(t)$ denote the number of grams of salt in the basin after t minutes.

Using the $\binom{\text{rate}}{\text{in}} - \binom{\text{rate}}{\text{out}}$ principle, write a differential equation that correctly determines x as a function of t and then solve this differential equation. Assume that $x(0) = 0$ (in other words, you are starting with only pure water in the basin).

5a. Pure water is flowing into a tank at the rate of 4 liters/minute. Salt solution is being pumped out of this tank at 4 liters/minute.

Let $x(t)$ be the number of grams of salt in the tank after t minutes. At $t = 0$, there is one gram of salt dissolved in 2 liters of liquid. Find the formula for $x(t)$ by setting up and solving an appropriate first order differential equation.

b. Now, suppose we have two tanks. Pure water is flowing into tank **A** at the rate of 4 liters/minute. Salt solution is being pumped out of this tank and into tank **B** at 4 liters per minute. Salt solution is also being drained out of tank **B** at 4 liters per minute.

Let $x(t)$ be the number of grams of salt in tank **A** and let $y(t)$ be the number of grams of salt in tank **B** after t minutes. At $t = 0$, each tank contains 2 liters of liquid. Assume that $x(0) = 1$ and $y(0) = 0$. Note that the volume of liquid in each tank remains constant. Find the formula for $y(t)$ by setting up and solving an appropriate differential equation.

Assignment 6. *Matrix Algebra and Matrix Equations*

Read 9.1, 9.2, 9.3

You should be able to do the following problems:

Exercise 9.2 Problems 1 - 9

Exercise 9.3 Problems 1 - 8

Hand in the following problems:

1. Let's define matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{I}$ as follows:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Calculate $\mathbf{A} + \mathbf{B}$ and $3\mathbf{I} - \mathbf{A}$

2. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be defined exactly as in problem 1. Calculate each of the following matrix operations. If the operation is not defined, then state this explicitly.

$$\mathbf{AB} \qquad \mathbf{CA} \qquad \mathbf{A}^2 - 3\mathbf{A} + \mathbf{I} \qquad \mathbf{AC}$$

3. Each of the equations below are of the form $\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{b}}$. Solve for $\vec{\mathbf{X}} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. I recommend matrix reduction, although any legitimate method is acceptable. Express each answer in vector form. If there is no solution or if there are an infinite number of solutions, then state this explicitly.

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

4. Solve each of the following matrix equations. Same instructions as in problem 3.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ 5 & 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

5. Define $\mathbf{P}$, $\mathbf{X}$, $\mathbf{O}$ and $\mathbf{I}$ as follows:

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solve each of the following equations:

$$\mathbf{PX} = \mathbf{O} \qquad (\mathbf{P} - 2\mathbf{I})\mathbf{X} = \mathbf{O}$$

Assignment 7. *Matrix Inverses and Equations*

Read 9.2 - 9.3

You should be able to do the following problems:

Exercise 9.3 Problems 9 - 16

Hand in the following problems:

1. For each of the following matrices, find the inverse. If the inverse does not exist for any of these matrices, state this explicitly.

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

2. Find the inverse of the following three by three matrix:

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

3. Use your inverse from problem 2 to solve each of the following equations:

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

4. Find all values of λ for which the following matrix has no inverse:

$$\begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}$$

5. Define $\mathbf{A}$, $\mathbf{I}$, $\mathbf{O}$ and $\mathbf{X}$ as follows:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The equation $\mathbf{AX} = \lambda\mathbf{X}$ can be rewritten as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{O}$$

Find all solutions $\mathbf{X}$ for each of the following cases:

$$\lambda = 0 \qquad \lambda = 1 \qquad \lambda = -1$$

Assignment 8. *Eigenvalues, Eigenvectors and Diagonalization of Matrices*

Read 9.1 - 9.5

You should be able to do the following problems:

Exercise 9.5 Problems 1 - 16

Read the notes on *Diagonalization of Matrices* on the MA 345 website.

Hand in the following problems:

1. Is $\binom{1}{-1}$ an eigenvector of the following matrix? If so, find the corresponding eigenvalue.

$$\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix}$$

2. Find all eigenvalues and eigenvectors of each of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

3. Let $\mathbf{A}$ be defined as follows:

$$\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 6 & -2 \end{pmatrix}$$

a. Verify that $\mathbf{V}_1 = \binom{1}{2}$ is an eigenvector of $\mathbf{A}$ corresponding to some eigenvalue λ_1. Find λ_1.
b. Verify that $\lambda_2 = 2$ is an eigenvalue of $\mathbf{A}$. Let $\mathbf{V}_2$ be the corresponding eigenvector. Find $\mathbf{V}_2$. See if you can find the solution for $\mathbf{V}_2$ that has only integers in its coordinates.
c. Let $\mathbf{P}$ be the two by two matrix with $\mathbf{V}_1$ as the first column and $\mathbf{V}_2$ as its second column. Calculate the matrix product $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

Note: If you perform the calculations in parts (a), (b) and (c) correctly, then your answer for $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ *will be a* **diagonal** *matrix.*

4. The following matrix has an eigenvalue of $\lambda = 1$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -6 & 4 \end{pmatrix}$$

Find the eigenvector corresponding to eigenvalue $\lambda = 1$

Assignment 9. *Matrix Differential Equations*

Read Chapter 9. Also, read *Diagonalization of Matrices* posted on the MA 345 website.
You should be able to do the following problems:
Exercise 9.5 Problems 11 - 16, 26, 31 - 34, 45 - 46
Hand in the following problems:

1. Define $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -3 & 5 \end{pmatrix}$$

Use the eigenvectors of $\mathbf{A}$ to find a matrix $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{AP}$ is a *diagonal* matrix.

2. Let $\mathbf{A}$ and $\mathbf{P}$ be the matrices referred to in problem 1. Let $\vec{\mathbf{x}} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ Show how the substitution $\vec{\mathbf{v}} = \mathbf{P}^{-1}\vec{\mathbf{x}}$ can be used to find the general solution of the differential equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$.

3. One method of solving an equation of the form $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ is to combine all solutions of the form $e^{rt}\vec{\mathbf{u}}$ where r is an eigenvalue and $\vec{\mathbf{u}}$ is the corresponding eigenvector. Use this method to find the solution of $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$ where $\vec{\mathbf{x}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{A}$ is given by:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$$

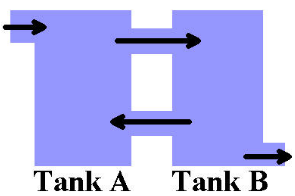
4. Let $\vec{\mathbf{x}}$ and $\mathbf{A}$ be defined as follows:

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Find the general solution of the differential equation $\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}$.

5. Two tanks, each holding 12 liters of liquid are interconnected by pipes with liquid flowing from tank A into tank B at 4 liters per minute and from tank B into tank A at 2 liter per minute. The liquid inside each tank is kept well stirred. Pure water flows into tank A at the rate of 2 liters per minute and the solution flows out of tank B at the rate of 2 liters per minute.

Let $x(t)$ be the number of grams of salt in tank A after t minutes.
Let $y(t)$ be the number of grams of salt in tank B after t minutes.
Let $\vec{\mathbf{X}} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. $\vec{\mathbf{X}}$ will solve the matrix differential equation $\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$. Find the matrix $\mathbf{A}$. You do not have to solve the differential equation for this problem.



Assignment 10. *Linear Differential Equations - Distinct Root Case*

Read 4.1 - 4.2
You should be able to do the following problems:
Exercise 4.2 Problems 1 - 20

Hand in the following problems:

For problems 1 - 4, find the general solution of each of the given equations

1.
$$y'' + 2y' - 7y = 0$$

2.
$$4y'' - 4y' - 3y = 0$$

3.
$$(D^3 + 3D^2 - 4D)y = 0$$

4.
$$y'' - 6y' - 16y = 0 \quad \text{where} \quad y(0) = 5 \quad \text{and} \quad y'(0) = 0$$

5.
$$\frac{d^2y}{dx^2} = 8\,\frac{dy}{dx} \quad \text{where} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 4$$

Assignment 11. *The Exponential Shift Theorem*

Read the notes on the Exponential Shift Theorem that are posted on the course website.
Hand in the following problems:

Problems 1 - 4. Use the Exponential Shift Theorem to calculate the following expressions:

1.
$$\left(D^2 - 4D + 5\right)\left(e^{2x}\sin x\right)$$

2.
$$\left(D^2 - 1\right)\left(e^x + xe^x + x^2e^x\right)$$

3.
$$D^3\left(e^{-x}x\right)$$

4.
$$(D-4)^2\left(e^{4x}x^4\right)$$

5. Solve the following differential equation by making the substitution $u = e^{-x}y$ and using the Exponential Shift Theorem.
$$(D-1)^2y = 0$$

Assignment 12. *Repeated and Complex Root Cases*

Read 4.3, 6.2 as well as the notes on Exponential Shift on the MA 345 website
You should be able to do the following problems:
Exercise 4.3 Problems 1 - 27, Exercise 6.2 Problems 1 - 6, 15 - 18
Hand in the following problems:

For problems 1 - 4, solve the given differential equation.

1. $$(4D^2 + 4D + 1)y = 0$$

2. $$(D^2 - 6D + 10)y = 0$$

3. $$(D - 2)(D^2 - 2D)y = 0$$

4. $$\frac{d^2y}{dt^2} + 2y = 0 \quad \text{where } y(0) = 0 \text{ and } y'(0) = 2$$

5. Use Euler's formula to simplify each of the following expressions:

a. $e^{i\pi/2}$ **b.** $e^{1+\pi i}$ **c** e^{4x+2ix}

Assignment 13. *Method of Undetermined Coefficients. Annihilators*

Read 4.4, 4.5, 6.3
You should be able to do the following problems:
Exercise 4.4 Problems 9 - 36, Exercise 4.5 Problems 3 - 40
Exercise 6.3 Problems 1 - 30
Hand in the following problems:

1. Find the annihilators for each of the following functions:

a. $x + e^{2x}$ **b.** xe^{2x} **c.** $e^x + e^{-x}$ **d.** $2\sin 2x + 4\cos 2x$

2. $$\frac{d^2y}{dx^2} + y = x^2 + x$$

3. $$(D - 3)^2 y = e^{2x}$$

4. $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\sin x$$

5. $$(D^2 - 4D)y = 16e^{4x}$$

Assignment 14. *Applications to Springs and Circuits*

Read 4.9, 4.10. 5.6 and 5.7
You should be able to do the following problems:
Exercise 4.9 Problems 1 - 10, Exercise 4.10 Problems 8 - 15
Hand in the following problems:

1. A spring with a spring constant of $k = 16$ N/m (Newtons per meter) is attached to a mass of $m = 2$ kg. Assume that the damping constant is $\beta = 0$. Let y be the displacement of the mass relative to the equilibrium position. Suppose the initial displacement is $y(0) = 4$ m and the initial velocity is $v(0) = 0$. Write a differential equation that determines y as a function of t. Solve the differential equation.

2. An object of mass 4 kg is attached to a spring with spring constant $k = 100$ N/m. Assume a resistive force due to friction with a damping constant of $\beta = 32$ kg/sec. If the object begins at the equilibrium position with an initial velocity of 3 m/sec, determine the position of the object as a function of time by solving an appropriate differential equation.

3. An object of mass 1 kg is attached to a spring with spring constant $k = 4$ N/m. There is a resistive force due to friction with a damping constant of $\beta = 4$ kg/sec. There is also a time dependent force acting on the object of $F(t) = 3te^{-2t}$. Let $y(t)$ denote the position of the object relative to the equilibrium position after t seconds. Assume that at $t = 0$, the object begins at its equilibrium position with a velocity of 0. Set up and solve the differential equation that determines $y(t)$.

4. An object of mass 2 kilograms is attached to a spring with a spring constant $k = 8$ newtons/meter. Assume that there is no damping force due to friction, but there is a time dependent force of $F(t) = 12\cos 2t$. Assume also that the object begins with an initial velocity of 0 and an initial displacement of 0. Determine the displacement of the object as a function of time by solving an appropriate differential equation.

5. An electric circuit has an electromotive force given by $\mathcal{E}(t) = 60\cos 5t$ volts, an inductor of 1 henry, a resistor of 6 ohms and a capacitor of 0.04 farads. Let $Q(t)$ be the charge on the capacitor. $Q(t)$ will solve the differential equation:

$$\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 25Q = 60\cos 5t$$

If $Q(t)$ satisfies the initial conditions $Q(0) = 0$ and $Q'(0) = 6$, find the formula for $Q(t)$.

Assignment 15. *Variation of Parameters*

Read 4.6 and 6.4
You should be able to do the following problems:
Exercise 4.6 Problems 1 - 18, Exercise 6.4 Problems 1 - 6
Hand in the following problems:

Solve each of the following by variation of parameters.

1.
$$y'' + y = \sin x \cos x$$

2.
$$(D^2 - D)y = \frac{e^{2x}}{e^{2x} + 1}$$

3.
$$y'' - 2y' + y = x^2 e^x$$

4. The Method of Variation of Parameters can be used to find the general solution of the following differential equation:

$$\left(D - \frac{1}{2}\right)^2 y = x^{-2} e^{\frac{x}{2}}$$

The solution has the form:

$$y = v_1(x) \cdot e^{\frac{x}{2}} + v_2(x) \cdot x e^{\frac{x}{2}}$$

where $v_1(x)$ and $v_2(x)$ are functions. Calculate the function $v_1(x)$

Assignment 16. *Introduction to Laplace Transforms*

Read 7.1, 7.2, 7.3, 7.4, 7.5
You should be able to do the following problems:
Exercise 7.2 Problems 1 - 20, Exercise 7.3 Problems 1 - 25
Exercise 7.4 Problems 1 - 30
Hand in the following problems:

1. The Laplace transform of a function $f(t)$ is defined to be: $\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$. Use this integral to calculate the Laplace transforms of each of the following functions. In each case, assume that s is large enough to guarantee convergence.

a) $t+1$ b) $(t+1)e^{-4t}$

2. Use a table of Laplace transforms as well as any relevant properties of Laplace transforms to find the following transforms:

a) $\mathcal{L}\left(e^{6t} - e^{-6t}\right)$ b) $\mathcal{L}\left(t^2 e^{-t}\right)$ c) $\mathcal{L}(\cos 2t)$ d) $\mathcal{L}\left(e^{3t}\cos 2t\right)$ e) $\mathcal{L}\left(e^{-t}\cosh t\right)$

3. Calculate the Laplace transform of the following function:

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ e^t & \text{if } 1 < t \end{cases}$$

4. Calculate the inverse Laplace transforms of each of the following:

a) $\frac{6}{(s-2)^2}$ b) $\frac{1}{(2s+1)^2}$ c) $\frac{8}{(s+2)(s+4)}$ d) $\frac{3}{(s^2+1)(s^2+4)}$ e) $\frac{1}{s^4+s^3}$

Assignment 17. *Using Laplace Transforms to Solve Differential Equations*

Read 7.5
You should be able to do the following problems:
Exercise 7.5 Problems 1 - 32
Hand in the following problems:

For problems 1 - 3, use the method of *Laplace transforms* to solve the differential equations:

1. $$y'' - 4y = -6e^t, \qquad y(0) = 3\ y'(0) = 0$$

2. $$y'' + 2y' = 4t + 6, \qquad y(0) = 0,\ y'(0) = 2$$

3. $$y'' - 2y' + 2y = 2, \qquad y(0) = 3,\ y'(0) = 0$$

4. Let $x = x(t)$ and $y = y(t)$ denote the solutions of the following system of differential equations:

$$\begin{aligned} x' &= x + 2y \\ y' &= 8x + y \end{aligned} \qquad \text{where } x(0) = 2 \text{ and } y(0) = 0$$

Use the method of Laplace transforms to solve this system of equations.

Assignment 18. *Additional Laplace Transform Topics - Step Functions*

Read 7.6, 7.7, 7.8 and 7.9

You should be able to do the following problems:

Exercise 7.6 Problems 1 - 36, Exercise 7.8 Problems 1 - 12, Exercise 7.9 Problems 13 - 20

Hand in the following problems:

1. Graph each of the following functions. The symbol $\mathcal{U}$ refers to the *unit step function.*

a. $f(t) = (\cos t) \cdot (\mathcal{U}(t) - \mathcal{U}(t-\pi))$

b. $g(t) = \mathcal{U}(t) - \mathcal{U}(t-1) + (2-t) \cdot (\mathcal{U}(t-1) - \mathcal{U}(t-2))$

2. Express the following $f(t)$ in terms of the unit step function $\mathcal{U}$ and then calculate $\mathcal{L}(f)$. You may use a table of Laplace transforms.

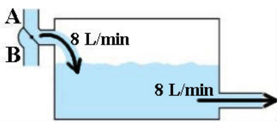
$$f(t) = \begin{cases} t & \text{when } 0 \leq t < 1 \\ e^{1-t} & \text{when } t \geq 1 \end{cases}$$

3. The electric charge $y(t)$ in an LC series circuit is governed by the initial value problem:

$$y''(t) + 4y(t) = 3\sin t - 3\sin t \cdot \mathcal{U}(t-2\pi), \qquad y(0) = 0,\ \ y'(0) = 0$$

where $\mathcal{U}$ is the *unit step function.* Determine the charge as a function of time t.

4. At $t = 0$, a tank contains 16 liters of a brine solution with 10 grams of salt dissolved in it. Then, we start draining fluid out at 8 liters per minute. For $0 \leq t \leq 2$, valve A is open and only pure water is being pumped in at 8 liters per minute. Then, for $t > 2$, valve A is closed and valve B is opened and salt solution containing $\frac{1}{2}$ grams of salt per liter is pumped in at 8 liters per minute.



a. Let $x(t)$ be the number of grams of salt in the tank after t minutes. Set up an appropriate differential equation for $x(t)$, making appropriate use of the unit step function $\mathcal{U}$.

b. Solve the differential equation using the method of Laplace transforms. Your final answer should be expressed in terms of the unit step function.

Assignment 19. *Series Solutions*

Read 8.2 and 8.3
You should be able to do the following problems:
Exercise 8.3 Problems 11 - 28
Hand in the following problems:

Problems 1 - 3 are to be solved by the method of *infinite series*. This means that you must obtain a solution of the form $\sum a_n x^n$. You must solve for a_n so that the sum solves the given differential equation. Try to express your final answer in summation notation.

1. $y'' - y = 0$ where $y(0) = y'(0) = 1$
2. $y'' - 2xy' - 2y = 0$ where $y(0) = 1,\ y'(0) = 0$
3. $(1 - 2x)y'' - 4y' = 0$ where $y(0) = 0,\ y'(0) = 2$
4. The following differential equation has a solution of the form $\sum a_n x^n$. Calculate only the coefficients a_2, a_3 and a_4.

$$\left(1 + x^2\right) y'' - 2y = 0 \quad \text{where } y(0) = y'(0) = 1$$

5. The following differential equation has a solution of the form $\sum a_n x^n$. Find the *recurrence relation* that gives a_{n+2} in terms of a_{n+1} and a_n. Express your answer in simplest form.

$$y'' + (1 + x)y' + y = 0$$

Assignment 20. *Orthogonal Trajectories*

This assignment is based on material covered in the last week of class, so none of the following problems will be collected.

Section 2.4 Problem 33. Find the orthogonal trajectories for each of the given families of curves:
a. $2x^2 + y^2 = k$ **b.** $y = kx^4$ **c.** $y = e^{kx}$ **d.** $y^2 = kx$