Spring Systems

$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = k_2 x_1 - k_2 x_2$$

Spring Systems

$$
\begin{aligned}
m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 \\
m_2 x_2'' &= k_2 x_1 - k_2 x_2
\end{aligned}
$$

$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$

$m_2 x_2'' = k_2 x_1 - k_2 x_2$

Take $m_1 = 8$, $m_2 = 1$, $k_1 = 12$, $k_2 = 4/3$

$$\begin{aligned} x_1'' &= -\frac{5}{3}x_1 + \frac{1}{6}x_2 \\ x_2'' &= \quad \frac{4}{3}x_1 - \frac{4}{3}x_2 \end{aligned}$$

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} -5/3 & 1/6 \\ 4/3 & -4/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{\mathbf{X}}'' = \mathbf{A}\vec{\mathbf{X}}$$

$$\mathbf{A} = \begin{pmatrix} -5/3 & 1/6 \\ 4/3 & -4/3 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1 \qquad \vec{\mathbf{u}}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\lambda_2 = -2 \qquad \vec{\mathbf{u}}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

Make the substitution $\vec{\mathbf{V}} = \mathbf{P}^{-1}\vec{\mathbf{X}}$

$$\vec{\mathbf{X}} = \mathbf{P}\vec{\mathbf{V}}$$

$$\vec{\mathbf{X}}'' = \mathbf{A}\vec{\mathbf{X}}$$

Substitute $\vec{\mathbf{X}} = \mathbf{P}\vec{\mathbf{V}}$

$$\mathbf{P}\vec{\mathbf{V}}'' = \mathbf{A}\mathbf{P}\vec{\mathbf{V}}$$

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$$\begin{pmatrix} v_1'' \\ v_2'' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\vec{\mathbf{V}}'' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\vec{\mathbf{V}}$$

$$\begin{pmatrix} v_1'' \\ v_2'' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} v_1'' \\ v_2'' \end{pmatrix} = \begin{pmatrix} -v_1 \\ -2v_2 \end{pmatrix}$$

$$v_1'' = -v_1 \qquad\qquad v_2'' = -2v_2$$

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The solution of $v_1'' + v_1 = 0$ is:

$$v_1 = a_1 \cos t + a_2 \sin t$$

The solution of $v_2'' + 2v_2 = 0$ is:

$$v_2 = b_1 \cos(\sqrt{2}\, t) + b_2 \sin(\sqrt{2}\, t)$$

$$v_1'' = -v_1 \qquad\qquad v_2'' = -2v_2$$

The solution of $v_1'' + v_1 = 0$ is:

$$v_1 = a_1 \cos t + a_2 \sin t$$

The solution of $v_2'' + 2v_2 = 0$ is:

$$v_2 = b_1 \cos(\sqrt{2}\, t) + b_2 \sin(\sqrt{2}\, t)$$

$$\vec{\mathbf{X}} = \mathbf{P}\vec{\mathbf{V}}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a_1 \cos t + a_2 \sin t \\ b_1 \cos(\sqrt{2}\, t) + b_2 \sin(\sqrt{2}\, t) \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a_1 \cos t + a_2 \sin t \\ b_1 \cos(\sqrt{2}\, t) + b_2 \sin(\sqrt{2}\, t) \end{pmatrix}$$

$$x_1 = a_1 \cos t + a_2 \sin t + b_1 \cos(\sqrt{2}\, t) + b_2 \sin(\sqrt{2}\, t)$$

$$x_2 = 4a_1 \cos t + 4a_2 \sin t - 2b_1 \cos(\sqrt{2}\, t) - 2b_2 \sin(\sqrt{2}\, t)$$