Exact Differential Equations
Dr. E. Jacobs

$$\frac{dy}{dx} = \frac{x - y}{x + y}$$

$$\frac{dy}{dx} = \frac{1 - y^2}{2xy - \sin y}$$

$$\frac{dy}{dx} = \frac{x - y}{x + y}$$

$$(x + y)dy = (x - y)dx$$

$$(y - x)dx + (x + y)dy = 0$$

This is now in the form:

$$M\,dx + N\,dy = 0$$

$$\frac{dy}{dx} = \frac{1 - y^2}{2xy - \sin y}$$

$$(2xy - \sin y)dy = (1 - y^2)dx$$

$$(y^2 - 1)dx + (2xy - \sin y)dy = 0$$

This is now in the form:

$$M\,dx + N\,dy = 0$$

Every equation of the form:

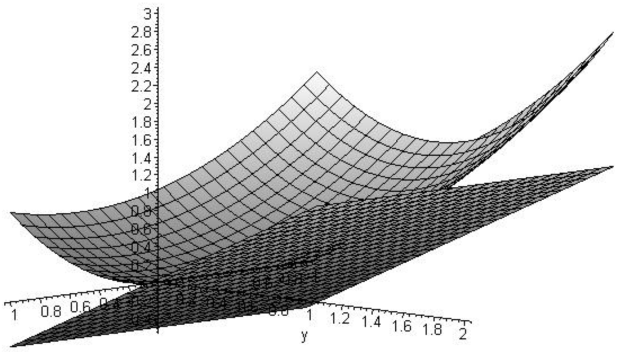
$$\frac{dy}{dx} = f(x, y)$$

can be rewritten in the form:

$$M\,dx + N\,dy = 0$$

where $M = M(x, y)$ and $N = N(x, y)$

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy$$



Example: If $F = xy - x^2$, find dF.

Solution:

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = (y - 2x)\, dx + x\, dy$$

Solve the following differential equation:

$$(y - 2x)\,dx + x\,dy = 0$$

Since $dF = (y - 2x)\,dx + x\,dy$
when $F = xy - x^2$, the equation becomes:

$$dF = 0$$

Integrate to get:

$$F = C \quad \text{where } C \text{ is some constant}$$

$$F = C$$

$$xy - x^2 = C$$

$$y = \frac{C + x^2}{x}$$

An expression of the form $M\,dx + N\,dy$ is said to be an *exact differential* if there is a function $F = F(x, y)$ such that

$$dF = M\,dx + N\,dy$$

If $dF = M\,dx + N\,dy$ then the equation $M\,dx + N\,dy = 0$ is called an exact differential equation.

Example: Find the general solution of:

$$(y - x)\,dx + (x + y)\,dy = 0$$

We can't solve by separation of variables.

Example: Find the general solution of:

$$(y - x)\,dx + (x + y)\,dy = 0$$

Try to find a function $F = F(x, y)$ such that $dF = (y - x)\,dx + (x + y)\,dy$

Example: Find the general solution of:

$$(y - x)\,dx + (x + y)\,dy = 0$$

Compare the two expressions:

$$dF = (y - x)\,dx + (x + y)\,dy$$

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

$$dF = (y - x)\,dx + (x + y)\,dy$$

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

$$\frac{\partial F}{\partial x} = y - x \quad \text{and} \quad \frac{\partial F}{\partial y} = x + y$$

(1)

$$\frac{\partial F}{\partial y} = x + y$$

(2)

$$F(x, y) = ????$$

$$\mathbf{(1)} \qquad \frac{\partial F}{\partial y} = x + y$$

$$\mathbf{(2)} \qquad F(x, y) = ????$$

$$F(x, y) = \int (x + y)\, dy \quad \text{(hold } x \text{ constant)}$$

$$= xy + \frac{1}{2}y^2 + C$$

If $F = xy + \frac{1}{2}y^2 + x^3$ then $\frac{\partial F}{\partial y} = x + y$

If $F = xy + \frac{1}{2}y^2 + \sin x$ then $\frac{\partial F}{\partial y} = x + y$

$$\frac{\partial F}{\partial y} = x + y \quad (\mathbf{1})$$

$$F(x, y) = ???? \quad (\mathbf{2})$$

$$F(x, y) = xy + \frac{1}{2}y^2 + g(x)$$

We have a second condition:

$$\frac{\partial F}{\partial x} = y - x$$

and this is what determines what $g(x)$ is. If $\frac{\partial F}{\partial x} = y - x$ and $F = xy + \frac{1}{2}y^2 + g(x)$ then:

$$\frac{\partial}{\partial x}\left(xy + \frac{1}{2}y^2 + g(x)\right) = y - x$$

$$\frac{\partial}{\partial x}\left(xy + \frac{1}{2}y^2 + g(x)\right) = y - x$$

$$y + g'(x) = y - x$$

$$g'(x) = -x$$

$$g(x) = -\frac{1}{2}x^2 + C_1$$

$$
\begin{aligned}
F(x,y) &= xy + \frac{1}{2}y^2 + g(x) \\
&= xy + \frac{1}{2}y^2 - \frac{1}{2}x^2 + C_1
\end{aligned}
$$

$$(y - x)\,dx + (x + y)\,dy = 0$$

$$d\left(xy + \frac{1}{2}y^2 - \frac{1}{2}x^2 + C_1\right) = 0$$

$$d\left(xy + \frac{1}{2}y^2 - \frac{1}{2}x^2\right) = 0$$

because $d(C_1) = 0$

$$xy + \frac{1}{2}y^2 - \frac{1}{2}x^2 = C$$

Example. Solve the following equation:

$$(y^2 - 1)dx + (2xy - \sin y)dy = 0$$

We can't solve by separation of variables. Let's try to find a function $F = F(x, y)$ so that

$$dF = (y^2 - 1)dx + (2xy - \sin y)dy$$

$$(y^2 - 1)dx + (2xy - \sin y)dy = 0$$

$$dF = (y^2 - 1)dx + (2xy - \sin y)dy$$

$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy$ so we want:

$$\frac{\partial F}{\partial x} = y^2 - 1 \quad \text{and} \quad \frac{\partial F}{\partial y} = 2xy - \sin y$$

This time let's start with $\frac{\partial F}{\partial x}$
If we know that

$$\frac{\partial F}{\partial x} = y^2 - 1$$

then we integrate with respect to x (holding y constant) to find F:

$$F = y^2 x - x + h(y)$$

$$F = y^2 x - x + h(y)$$

The condition $\frac{\partial F}{\partial y} = 2xy - \sin y$ will determine what $h(y)$ is.

$$\frac{\partial}{\partial y}\left(y^2 x - x + h(y)\right) = 2xy - \sin y$$

$$2xy + h'(y) = 2xy - \sin y$$

$$h'(y) = -\sin y$$

$$h(y) = \cos y + C_1$$

$$F = y^2x - x + h(y)$$

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$$2xy + h'(y) = 2xy - \sin y$$

$$h'(y) = -\sin y$$

$$h(y) = \cos y$$

$$
\begin{aligned}
F &= y^2 x - x + h(y) \\
&= y^2 x - x + \cos y
\end{aligned}
$$

So the solution of

$$(y^2 - 1)dx + (2xy - \sin y)dy = 0$$

is:

$$y^2 x - x + \cos y = C$$

$$
\begin{aligned}
F &= y^2x - x + h(y) \\
&= y^2x - x + \cos y
\end{aligned}
$$

So the solution of

$$
(y^2 - 1)dx + (2xy - \sin y)dy = 0
$$

is:

$$
y^2x - x + \cos y = C
$$

$$
x = \frac{C - \cos y}{y^2 - 1}
$$

Example: Solve the equation:

$$\left(y + \frac{2}{x}\right) dx + x\, dy = 0 \quad \text{where } y(1) = 0$$

We will look for F such that

$$\frac{\partial F}{\partial x} = y + \frac{2}{x} \;\; \text{and} \;\; \frac{\partial F}{\partial y} = x$$

Let's start with $\frac{\partial F}{\partial x} = y + \frac{2}{x}$.

$$F = \int \left(y + \frac{2}{x} \right) \, dx = xy + 2 \ln x + h(y)$$

$h(y)$ is determined by the condition $\frac{\partial F}{\partial y} = x$

$$\frac{\partial}{\partial y} \left(xy + 2 \ln x + h(y) \right) = x$$

$$x + h'(y) = x$$

$$h'(y) = 0$$

$$h(y) = C_1 \quad \text{where } C_1 \text{ is some constant}$$

$$
\begin{aligned}
F &= xy + 2\ln x + h(y) \\
&= xy + 2\ln x + C_1
\end{aligned}
$$

$$
\begin{aligned}
F &= xy + 2\ln x + h(y) \\
&= xy + 2\ln x
\end{aligned}
$$

$F = xy + 2\ln x$
The solution of $\left(y + \frac{2}{x}\right)\,dx + x\,dy = 0$ is:

$$xy + 2\ln x = C$$

$y(1) = 0$ implies that $C = 0$ so:

$$xy + 2\ln x = 0$$

$$y = -\frac{2\ln x}{x}$$

$$M\,dx + N\,dy = 0$$

For each of these problems, we have been solving $M\,dx + N\,dy = 0$ by looking for a function $F = F(x, y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

What if there were no such function F? How could we tell?

$$\left(2y + \frac{1}{x}\right) dx + x\, dy = 0$$

$$\frac{\partial F}{\partial x} = 2y + \frac{1}{x} \quad \text{and} \quad \frac{\partial F}{\partial y} = x$$

If $\frac{\partial F}{\partial x} = 2y + \frac{1}{x}$ then $F = 2xy + \ln x + h(y)$. But then the condition that $\frac{\partial F}{\partial y} = x$ would imply that

$$\frac{\partial}{\partial y}\left(2xy + \ln x + h(y)\right) = x$$

$$\frac{\partial}{\partial y}\left(2xy + \ln x + h(y)\right) = x$$

$$2x + h'(y) = x$$

$$h'(y) = -x$$

$$\frac{\partial}{\partial y}\left(2xy + \ln x + h(y)\right) = x$$

$$2x + h'(y) = x$$

$$h'(y) = -x$$

CONTRADICTION!

Therefore, there is no such function F such that $dF = \left(2y + \frac{1}{x}\right) dx + x\, dy$.

Testing for exactness

If there is a function F such that $dF = M\,dx + N\,dy$ then:

$$M = \frac{\partial F}{\partial x} \quad \text{and} \quad N = \frac{\partial F}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial N}{\partial x}$$

If $M\,dx + N\,dy$ is exact, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then $M\,dx + N\,dy$ is not exact.

That is, there is no F such that

$$dF = M\,dx + N\,dy$$

To illustrate this test, let's go back to the equation

$$\left(2y + \frac{1}{x}\right) dx + x\, dy = 0$$

where $M = 2y + \frac{1}{x}$ and $N = x$.

$$\frac{\partial M}{\partial y} = 2 \text{ but } \frac{\partial N}{\partial x} = 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, we would know not to waste our time trying to find F such that $dF = \left(2y + \frac{1}{x}\right) dx + x\, dy$.

Integrating Factors
Dr. E. Jacobs

Methods so far:

Separation of variables
Exact differential

How can we solve $\left(2y + \frac{1}{x}\right) dx + x\,dy = 0$ if the differential equation is not exact?

How can we solve $\left(2y + \frac{1}{x}\right) dx + x\, dy = 0$ if the differential equation is not exact?

Integrating Factor

Definition of Integrating Factor

Suppose

$$M\,dx + N\,dy$$

is not an exact differential, but there is a function $\mu = \mu(x, y)$ such that

$$M\mu\,dx + N\mu\,dy$$

is an exact differential, then μ is called an integrating factor.

$$\left(2y + \frac{1}{x}\right) dx + x\, dy = 0$$

Multiply both sides by x

$$(2xy + 1)dx + x^2 dy = 0$$

Now, $M = 2xy + 1$ and $N = x^2$ and if we check the partial derivatives,

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

and now we have an exact differential equation.

How do we find the integrating factor?

$$M\,dx + N\,dy \qquad \text{Not Exact}$$

$$\mu M\,dx + \mu N\,dy \qquad \text{Exact}$$

How do we find the integrating factor?

$$M\,dx + N\,dy \qquad \text{Not Exact}$$

$$\mu M\,dx + \mu N\,dy \qquad \text{Exact}$$

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}$$

Find $\mu = \mu(x, y)$

However, if $\mu = \mu(x)$ the equation simplifies

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}$$

$$M \cdot 0 + \mu\frac{\partial M}{\partial y} = N\frac{d\mu}{dx} + \mu\frac{\partial N}{\partial x}$$

$$M \cdot 0 + \mu \frac{\partial M}{\partial y} = N \frac{d\mu}{dx} + \mu \frac{\partial N}{\partial x}$$

$$N \frac{d\mu}{dx} = \mu \frac{\partial M}{\partial y} - \mu \frac{\partial N}{\partial x}$$

$$\frac{1}{\mu} \frac{d\mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$\frac{d}{dx} (\ln \mu(x)) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$(xy+1)dx + x^2dy = 0$$

The first thing we do is check for exactness.

$$M = xy + 1 \quad \text{and} \quad N = x^2$$

$$\frac{\partial M}{\partial y} = x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x$$

This is not an exact differential.

$$(xy+1)dx + x^2 dy = 0$$

Is there an integrating factor $\mu = \mu(x)$?

$$M = xy + 1 \quad \text{and} \quad N = x^2$$

$$\frac{\partial M}{\partial y} = x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x$$

$$\frac{d}{dx}(\ln \mu(x)) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x - 2x}{x^2} = -\frac{1}{x}$$

$$\frac{d}{dx}(\ln \mu(x)) = -\frac{1}{x}$$

$$\ln \mu = -\ln x = \ln\left(x^{-1}\right)$$

$$\mu = x^{-1}$$

Multiply both sides of the differential equation and we should get an exact differential

$$(xy + 1)dx + x^2 dy = 0$$

$$(xy + 1)x^{-1}\, dx + x^2 \cdot x^{-1}\, dy = 0 \cdot x^{-1}$$

$$(y + x^{-1})\, dx + x\, dy = 0$$

$$(y + x^{-1})\,dx + x\,dy = 0$$

$$M = y + x^{-1} \quad \text{and} \quad N = x$$

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

Therefore, we can find a function $F(x, y)$ so that:

$$dF = (y + x^{-1})\,dx + x\,dy = 0$$

$$dF = (y + x^{-1})\,dx + x\,dy = 0$$

$$F = xy + \ln x$$

Therefore, the solution is:

$$xy + \ln x = C$$

$$y = \frac{C - \ln x}{x}$$

Example: Solve the following equation:

$$1\,dx + (x\tan y - \cos y)dy = 0$$

Start by checking for exactness.
$M = 1$ and $N = x\tan y - \cos y$.

$$\frac{\partial M}{\partial y} = 0 \quad \text{and} \quad \frac{\partial N}{\partial x} = \tan y$$

This isn't an exact differential equation.

$$1\,dx + (x\tan y - \cos y)dy = 0$$

Maybe there's an integrating factor $\mu(x)$

$$\frac{d}{dx}(\ln \mu(x)) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{0 - \tan y}{x\tan y - \cos y}$$

$$1\,dx + (x\tan y - \cos y)dy = 0$$

Maybe there's an integrating factor $\mu(x)$

$$\frac{d}{dx}(\ln \mu(x)) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{0 - \tan y}{x\tan y - \cos y}$$

Contradiction!

There is no integrating factor $\mu(x)$

$$1\,dx + (x\tan y - \cos y)dy = 0$$

Maybe there's an integrating factor $\mu(y)$

To find the integrating factor μ we must solve

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

$$M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}$$

If $\mu = \mu(y)$ then simplifies to:

$$M\frac{d\mu}{dy} + \mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x}$$

$$M\frac{d\mu}{dy} + \mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x}$$

After some rearranging of terms, we get:

$$\frac{1}{\mu}\frac{d\mu}{dy} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

or, equivalently,

$$\frac{d}{dy}(\ln\mu(y)) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

$$1\,dx + (x\tan y - \cos y)dy = 0$$

If there is an integrating factor $\mu = \mu(y)$ then:

$$\frac{d}{dy}(\ln \mu(y)) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\tan y - 0}{1}$$

$$\frac{d}{dy}(\ln \mu(y)) = \tan y$$

$$\begin{aligned}\ln \mu &= \int \tan y \, dy \\ &= -\ln(\cos(y)) \\ &= \ln\left(\sec y\right)\end{aligned}$$

If $\ln \mu = \ln \sec y$ then:

$$\mu = \sec y$$

$$\mu = \sec y$$

$$1\,dx + (x \tan y - \cos y)dy = 0$$

$$\sec y\,dx + (x \sec y \tan y - \sec y \cos y)dy = 0$$

$$\sec y\,dx + (x \sec y \tan y - 1)dy = 0$$

$$\frac{\partial M}{\partial y} = \sec y \tan y \quad \text{and} \quad \frac{\partial N}{\partial x} = \sec y \tan y$$

This must be an exact differential equation.

$$\sec y\, dx + (x \sec y \tan y - 1)dy = 0$$

$$dF = \sec y\, dx + (x \sec y \tan y - 1)dy = 0$$

$$\frac{\partial F}{\partial x} = \sec y \quad \text{and} \quad \frac{\partial F}{\partial x} = x \sec y \tan y - 1$$

$$F = x \sec y - y$$

Therefore, the solution to the differential equation is:

$$x \sec y - y = C$$

Summary of Procedure

To solve the equation $M\,dx + N\,dy = 0$ go through the following steps:

1. You might as well check to see if you can solve by separation of variables first.

2. Check if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
If these are the same, solve for a function F so that $dF = M\,dx + N\,dy$ and the solution of the differential equation is $F(x, y) = C$.

3. If the equation is not exact, try to find an integrating factor $\mu = \mu(x)$ If the expression $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ does not depend on the variable y at all, then the following equation can be used to get the integrating factor:

$$\frac{d}{dx}\left(\ln \mu(x)\right) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

Multiply both sides by $\mu(x)$ and you will get an exact differential equation.

4. If there is no integrating factor depending on x alone, then try to find $\mu = \mu(y)$ If the expression $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ does not depend on the variable x at all, then the following equation can be used to get the integrating factor:

$$\frac{d}{dy}(\ln \mu(y)) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

Multiply both sides by $\mu(y)$ and you will get an exact differential equation.

5. If the integrating factor μ does not depend on x alone and it does not depend on y alone, then μ will depend on both x and y and it may be impossible to find. In this case, there may be some special variable substitution that can be used to solve $M\,dx + N\,dy = 0$.