Differential Equations - Special Substitutions

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$$\left(y^3 - x^2 y\right)\, dx + x^3\, dy = 0 \quad \text{where } y(1) = 1$$

Can we solve using separation of variables?

$$\left(y^3 - x^2 y\right)\, dx + x^3\, dy = 0 \quad \text{where } y(1) = 1$$

$$y^3\, dx + x^3\, dy = x^2 y\, dx$$

$$\left(y^3 - x^2 y\right)\, dx = -x^3\, dy$$

$$-x^2 y\, dx + x^3\, dy = -y^3\, dx$$

$$\frac{dy}{dx} = \frac{x^2 y - y^3}{x^3}$$

$$\left(y^3 - x^2 y\right)\, dx + x^3\, dy = 0$$

Is this an exact differential?

$$M = y^3 - x^2 y \qquad\qquad N = x^3$$

$$\frac{\partial M}{\partial y} = 3y^2 - x^2 \qquad\qquad \frac{\partial N}{\partial x} = 3x^2$$

$$\left(y^3 - x^2 y\right)\, dx + x^3\, dy = 0$$

Is there an integrating factor $\mu(x)$?

$$\frac{d}{dx}\left(\ln \mu(x)\right) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3y^2 - 4x^2}{x^3}$$

$$\left(y^3 - x^2 y\right)\, dx + x^3\, dy = 0$$

Is there an integrating factor $\mu(y)$?

$$\frac{d}{dy}\left(\ln \mu(y)\right) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{4x^2 - 3y^2}{y^3 - x^2 y}$$

$$\left(y^3 - x^2 y\right)\, dx + x^3\, dy = 0$$

$$y^3\, dx + x^3\, dy = x^2 y\, dx$$

$$\left(y^3 - x^2 y\right)\, dx = -x^3\, dy$$

$$-x^2 y\, dx + x^3\, dy = -y^3\, dx$$

$$\frac{dy}{dx} = \frac{x^2 y - y^3}{x^3}$$

$$\frac{dy}{dx} = \frac{x^2y - y^3}{x^3} = \frac{y}{x} - \left(\frac{y}{x}\right)^3$$

This is now in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Let $v = \frac{y}{x}$

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^3$$

If $v = \dfrac{y}{x}$ then $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = v - v^3$$

$$v + x\frac{dv}{dx} = v - v^3$$

$$x\frac{dv}{dx} = -v^3$$

$$v^{-3}\,dv = -\frac{1}{x}\,dx$$

$$v + x\frac{dv}{dx} = v - v^3$$

$$x\frac{dv}{dx} = -v^3$$

$$\int v^{-3}\,dv = -\int \frac{1}{x}\,dx$$

$$-\frac{1}{2}v^{-2} = -\ln|x| + C$$

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$$v^{-2} = 2\ln|x| - 2C = \ln\left(x^2\right) - 2C$$

If $y = 1$ when $x = 1$ then $v = \frac{y}{x} = 1$ also and so $C = -\frac{1}{2}$

$$v^{-2} = 1 + \ln\left(x^2\right)$$

$$v^{-2} = 1 + \ln\left(x^2\right)$$

$$v^2 = \frac{1}{1 + \ln\left(x^2\right)}$$

$$v = \frac{1}{\sqrt{1 + \ln\left(x^2\right)}}$$

If $v = \frac{y}{x}$ then

$$y = x \cdot v = \frac{x}{\sqrt{1 + \ln\left(x^2\right)}}$$

In general, if we let $v = \frac{y}{x}$ then

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \text{becomes} \quad v + x\frac{dv}{dx} = f(v)$$

$$x\frac{dv}{dx} = f(v) - v$$

$$\frac{dv}{f(v) - v} = \frac{dx}{x}$$

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$$

$$\frac{dy}{dx} + \frac{y}{x} = 2 \qquad \text{where } y(1) = 2$$

$$\frac{dy}{dx} = 2 - \frac{y}{x}$$

This is now in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ so let $v = \frac{y}{x}$

$$\frac{dy}{dx} = 2 - \frac{y}{x} = 2 - v$$

If $v = \frac{y}{x}$ then $\quad \frac{d}{dx}(vx) = 2 - v$

$$v + x\frac{dv}{dx} = 2 - v$$

$$x\frac{dv}{dx} = 2 - 2v$$

$$\frac{1}{1-v}dv = \frac{2}{x}\,dx$$

$$\int \frac{1}{1-v} dv = \int \frac{2}{x}\, dx$$

$$-\ln|1-v| = 2\ln|x| + C = \ln\left(x^2\right) + C$$

$$-\ln|1-v| - \ln\left(x^2\right) = C$$

$$\ln|1-v| + \ln\left(x^2\right) = -C$$

$$\ln\left(x^2|1-v|\right) = -C$$

$$x^2|1-v| = e^{-C}$$

$$|1-v| = \frac{e^{-C}}{x^2}$$

$$|1 - v| = \frac{e^{-C}}{x^2}$$

$$1 - v = \frac{\pm e^{-C}}{x^2} = \frac{a}{x^2}$$

$$v = 1 - \frac{a}{x^2}$$

$$v = 1 - \frac{a}{x^2}$$

Since $v = \frac{y}{x}$

$$y = x \cdot v = x\left(1 - \frac{a}{x^2}\right) = x - \frac{a}{x}$$

$y(1) = 2$ implies that $a = -1$

$$y = x + \frac{1}{x}$$

$$a_1 \frac{dy}{dx} + a_0 y = f(x)$$

First order linear differential equation.

Example:

$$\frac{dy}{dx} + \frac{y}{x} = 2$$

$$a_1 = 1 \qquad a_0 = \frac{1}{x} \qquad f(x) = 2$$

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

Second order linear differential equation.