Differential Equations
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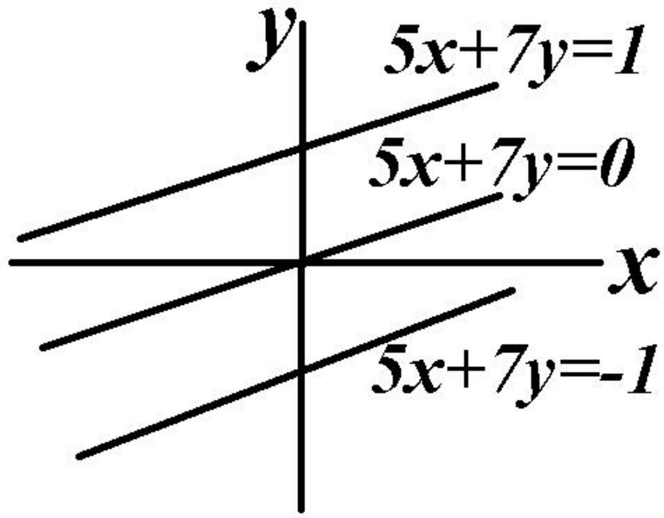
Topic for Today: Linear Equations

The following expression is called a *linear combination* of the variables x and y

$$ax + by$$

An equation of the form $ax + by = C$ describes a straight line



Linear combination of x, y and z

$$2x + (-5)y + 9z$$

or equivalently,

$$2x - 5y + 9z$$

More generally, a linear combination of the variables x_1, x_2, x_3, ..., x_n is an expression of the form:

$$c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n$$

More generally, a linear combination of the variables x_1, x_2, x_3, …, x_n is an expression of the form:

$$c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n$$

$$\sum_{k=1}^{n} c_k x_k$$

Linear Combinations of Vectors

Let $\vec{\mathbf{u}} = \langle 2, 3, 4 \rangle = 2\vec{\mathbf{i}} + 3\vec{\mathbf{j}} + 4\vec{\mathbf{k}}$
Let $\vec{\mathbf{v}} = \langle 1, 0, 5 \rangle = 1\vec{\mathbf{i}} + 0\vec{\mathbf{j}} + 5\vec{\mathbf{k}}$

Both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and linear combinations of the vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$ and $\vec{\mathbf{k}}$

Linear Combinations of Functions

Let $\phi(x) = e^{x^2}$ and $\psi(x) = \sin x$

The expression

$$5\phi + 7\psi$$

is a linear combination of ϕ and ψ and represents the function:

$$5e^{x^2} + 7\sin x$$

Linear combination of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$2\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y$$

Linear differential equation

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$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 2$$

The coefficients of a linear differential equation could be functions of the independent variable.

$$e^x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} + x^2 y = 0$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

We will begin with first order linear differential equations:

$$a_1 \frac{dy}{dx} + a_0 y = f(x)$$

$$a_1\frac{dy}{dx} + a_0 y = f(x)$$

$$\frac{dy}{dx} + \frac{a_0}{a_1}y = \frac{f(x)}{a_1}$$

Let $P(x) = \frac{a_0}{a_1}$ and let $Q(x) = \frac{f(x)}{a_1}$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

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$$(P(x)y - Q(x))dx + 1\,dy = 0$$

$$M = P(x)y - Q(x) \qquad N = 1$$

Is there an integrating factor $\mu(x)$?

$$\frac{d}{dx}(\ln \mu(x)) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = P(x)$$

$$\frac{d}{dx}(\ln \mu(x)) = P(x)$$

$$\ln \mu(x) = \int P(x)\, dx$$

$$\mu(x) = e^{\int P(x)\, dx}$$

$$\frac{d}{dx}(\ln \mu(x)) = P(x)$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = P(x)$$

$$\frac{d\mu}{dx} = \mu P(x)$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Multiply both sides of the equation by μ

$$\mu\frac{dy}{dx} + \mu P(x)y = \mu Q(x)$$

$$\mu\frac{dy}{dx} + \frac{d\mu}{dx} \cdot y = \mu Q(x)$$

$$\frac{d}{dx}\left(\mu y\right) = \mu Q(x)$$

$$\frac{dy}{dx} + \frac{y}{x} = 2$$

This is a linear differential equation with $P(x) = \frac{1}{x}$ and $Q(x) = 2$. The integrating factor is:

$$\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = 2$$

$$x\left(\frac{dy}{dx} + \frac{1}{x} \cdot y\right) = 2x$$

$$x\frac{dy}{dx} + y = 2x$$

$$\frac{d}{dx}(xy) = 2x$$

$$\frac{d}{dx}(xy) = 2x$$

$$xy = x^2 + C$$

$$y = x + \frac{C}{x}$$

$$\mu = e^{\int \frac{1}{x}\, dx} = e^{\ln x} = x$$

More generally,

$$\mu = e^{\ln|x|+C} = e^C|x| = \pm e^C x$$

$$\text{Let } a = \pm e^C$$

$$\mu = ax$$

Be careful with exponential properties:

$$e^{\ln u} = u$$

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$$e^{\ln u} = u$$

$$e^{n \ln u} = e^{\ln(u^n)} = u^n$$

$$\frac{dy}{dx} - \frac{1}{2x}y = 1$$

This is in the form $\frac{dy}{dx} + P(x)y = Q(x)$ where

$$P(x) = -\frac{1}{2x}$$

Calculate the integrating factor:

$$\mu = e^{\int \frac{-1}{2x}\, dx} = e^{-\frac{1}{2}\ln x} = x^{-1/2}$$

$$\frac{dy}{dx} - \frac{1}{2x} y = 1$$

$$x^{-1/2} \left(\frac{dy}{dx} - \frac{1}{2x} y \right) = x^{-1/2}$$

$$x^{-1/2} \frac{dy}{dx} - \frac{1}{2} x^{-3/2} y = x^{-1/2}$$

$$\frac{d}{dx} \left(x^{-1/2} y \right) = x^{-1/2}$$

$$\frac{d}{dx}\left(x^{-1/2}y\right) = x^{-1/2}$$

$$x^{-1/2}y = \int x^{-1/2}\,dx = 2x^{1/2} + C$$

$$y = 2x + Cx^{1/2}$$