Differential Equations

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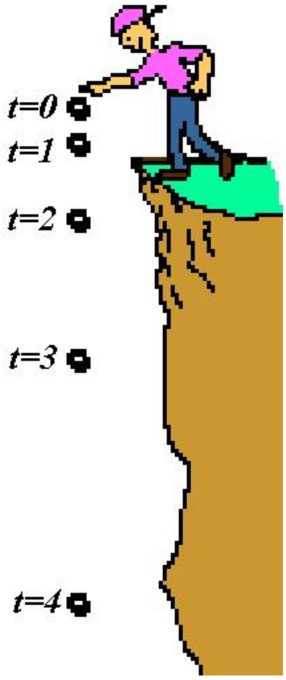
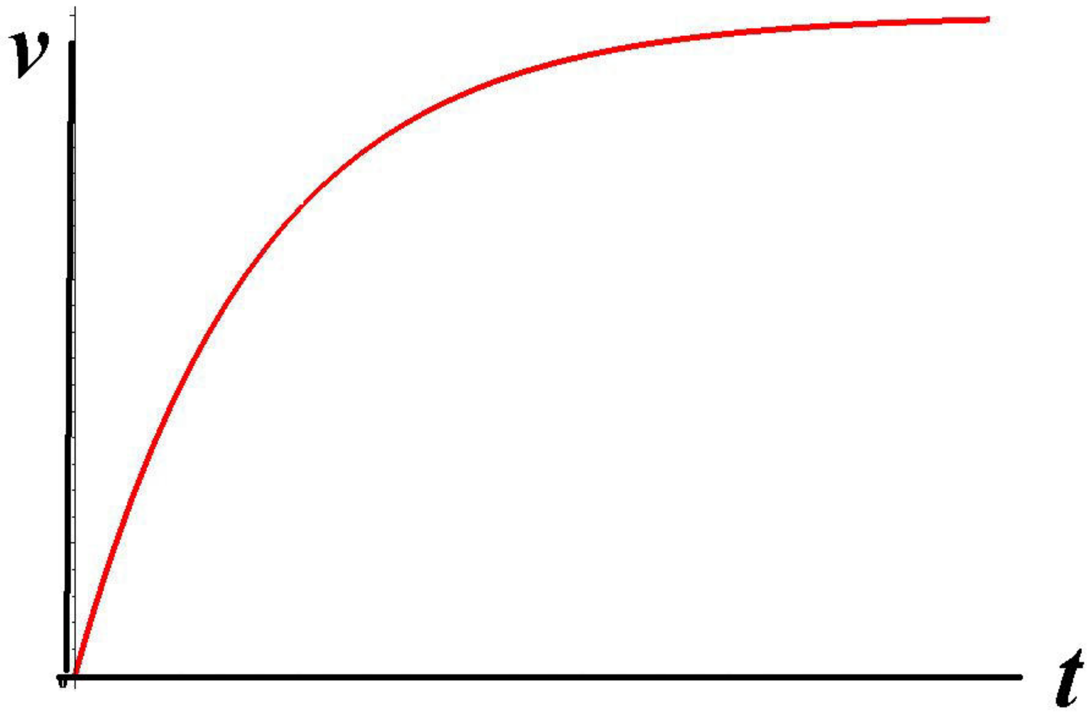
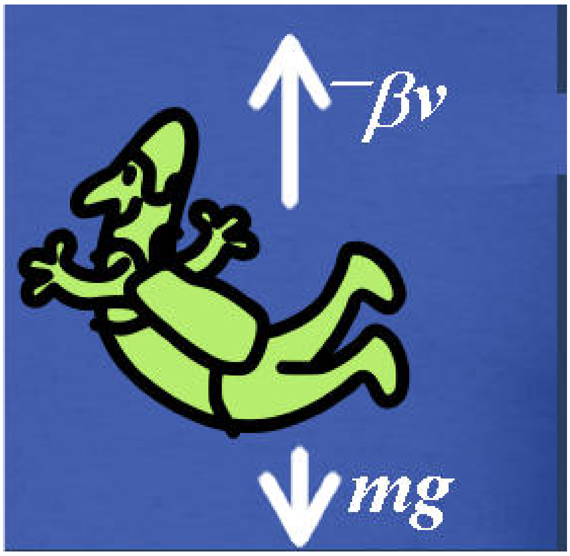
The Skydiver Problem

The velocity of a falling object increases with time

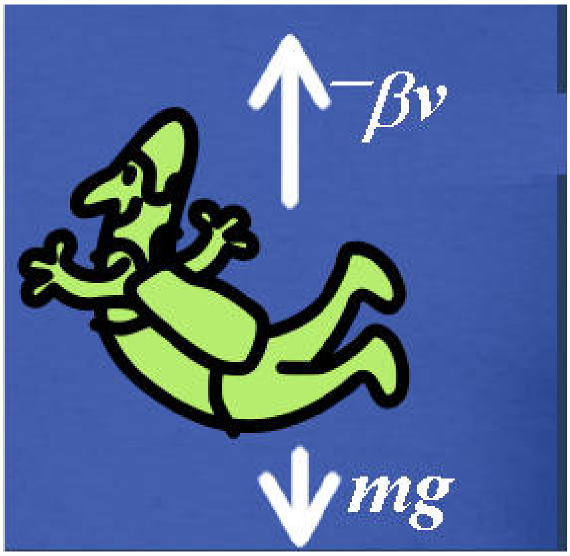



v
t

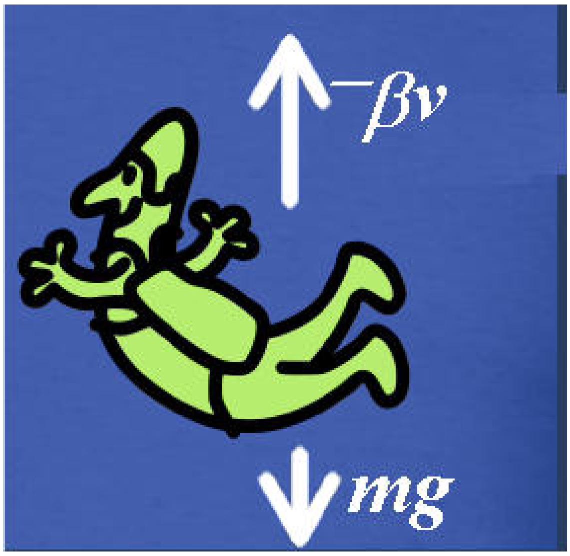
$$ma = F$$



$$ma = F = mg - \beta v$$



$$m\frac{dv}{dt} = mg - \beta v$$



$$m\frac{dv}{dt} = mg - \beta v$$

$$\frac{dv}{dt} + \frac{\beta}{m}v = g$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\text{Let } \mu = e^{\int P(x)\,dx}$$

$$\mu \frac{dy}{dx} + \mu P y = \mu Q$$

$$\frac{d}{dx}(\mu y) = \mu Q$$

$$\frac{dv}{dt} + \frac{\beta}{m}v = g$$

This is in the form $\frac{dv}{dt} + P(t)v = Q(t)$ with $P(t) = \frac{\beta}{m}$ and $Q(t) = g$

$$\mu = e^{\int P(t)\, dt} = e^{\int \frac{\beta}{m}\, dt} = e^{\frac{\beta}{m}t}$$

$$\frac{dv}{dt} + \frac{\beta}{m} v = g$$

$$e^{\frac{\beta t}{m}} \frac{dv}{dt} + \frac{\beta}{m} e^{\frac{\beta t}{m}} v = g e^{\frac{\beta t}{m}}$$

$$\frac{d}{dt}\left(e^{\frac{\beta t}{m}} v\right) = g e^{\frac{\beta t}{m}}$$

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$$e^{\frac{\beta t}{m}} v = \int g e^{\frac{\beta t}{m}}\, dt = \frac{mg}{\beta} e^{\frac{\beta t}{m}} + C$$

$$v = \frac{mg}{\beta} + C e^{-\frac{\beta t}{m}}$$

$$v = \frac{mg}{\beta} + Ce^{-\frac{\beta t}{m}}$$

If $v = 0$ when $t = 0$ then $C = -\frac{mg}{\beta}$

$$v = \frac{mg}{\beta} - \frac{mg}{\beta} e^{-\frac{\beta t}{m}}$$

$$v = \frac{mg}{\beta} \left(1 - e^{-\frac{\beta t}{m}}\right)$$

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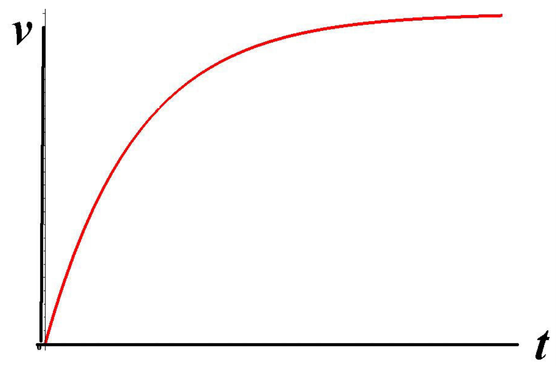

$$\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{mg}{\beta}\left(1 - e^{-\frac{\beta t}{m}}\right) = \frac{mg}{\beta}$$

$$m\frac{dv}{dt} = mg - \beta v$$

$$\frac{m\,dv}{mg - \beta v} = dt$$

$$\int \frac{m\,dv}{mg - \beta v} = \int dt$$

$$\int \frac{m\,dv}{mg - \beta v} = \int dt$$

Let $u = mg - \beta v \qquad\qquad du = -\beta\,dv$

$$-\frac{m}{\beta} \int \frac{1}{u}\,du = \int dy$$

$$\int \frac{1}{u}\,du = -\frac{\beta}{m} \int dt$$

$$\ln u = -\frac{\beta}{m}t + C$$

$$\ln(mg - \beta v) = -\frac{\beta}{m}t + C$$

Since $v = 0$ when $t = 0$, we get $C = \ln(mg)$

$$\ln(mg - \beta v) = -\frac{\beta}{m}t + \ln(mg)$$

$$\ln(mg - \beta v) = -\frac{\beta t}{m} + \ln(mg)$$

$$mg - \beta v = e^{-\frac{\beta t}{m} + \ln(mg)} = mge^{-\frac{\beta t}{m}}$$

$$v = \frac{mg}{\beta}\left(1 - e^{-\frac{\beta t}{m}}\right)$$