Applications of 1^{st} Order Equations

Dr. Elliott Jacobs

$$\frac{dy}{dt} + P(t)y = Q(t)$$

$$\text{Let } \mu = e^{\int P(t)\,dt}$$

$$\mu \frac{dy}{dt} + \mu P y = \mu Q$$

$$\frac{d}{dt}(\mu y) = \mu Q$$

A refrigerator is kept at 2 degrees Centigrade. At $t = 0$, a cup of coffee is placed inside the refrigerator. The coffee starts off at 66 degrees. After 1 minute, it has cooled down to 50 degrees.

Find the temperature after t minutes.

Newton's Law of Cooling:

The rate at which the temperature changes is proportional to the temperature difference between the object and its environment.

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$$\frac{dy}{dt} = k(y - 2)$$

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$$\frac{dy}{dt} = -k(y - 2)$$

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$$\frac{dy}{dt} + ky = 2k$$

This is now in the form $\frac{dy}{dt} + P(t)y = Q(t)$.

$$\mu = e^{\int k \, dt} = e^{kt}$$

$$\frac{dy}{dt} + ky = 2k$$

$$e^{kt}\frac{dy}{dt} + ke^{kt}y = 2ke^{kt}$$

$$\frac{d}{dt}\left(e^{kt}y\right) = 2ke^{kt}$$

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$$e^{kt}y = 2e^{kt} + C$$

The coffee starts off at 66 degrees. After 1 minute, it has cooled down to 50 degrees.

$e^0 \cdot 66 = 2e^0 + C$ implies $C = 64$

$$y = 2 + 64e^{-kt}$$

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After 1 minute, the coffee cooled down to 50 degrees.

$$50 = 2 + 64e^{-k}$$

$$\frac{3}{4} = e^{-k}$$

$$e^{-k} = \frac{3}{4}$$

$$y = 2 + 64e^{-kt} = 2 + 64\left(e^{-k}\right)^t = 2 + 64\left(\frac{3}{4}\right)^t$$

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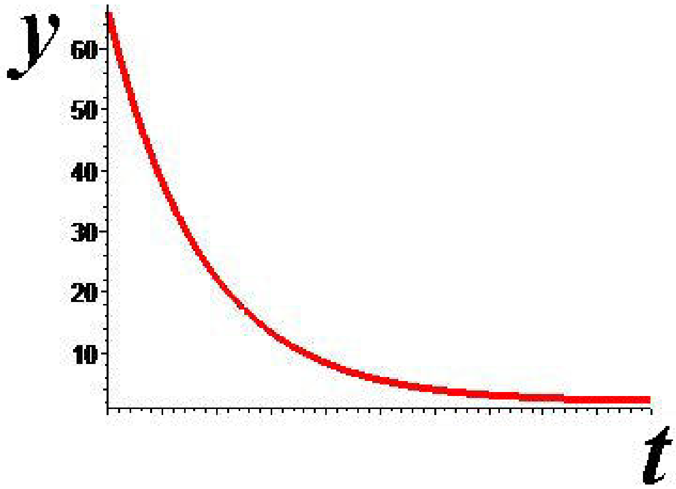
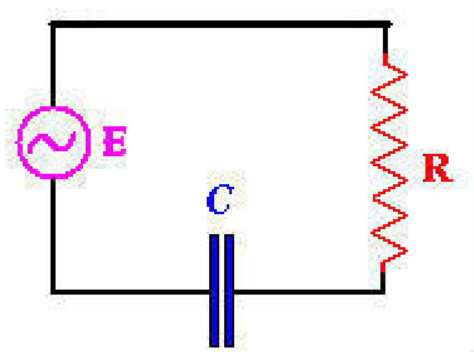
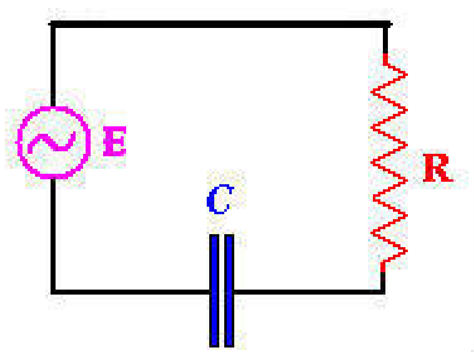


Sum of the voltage drops=Electromotive force



Sum of the voltage drops=Electromotive force

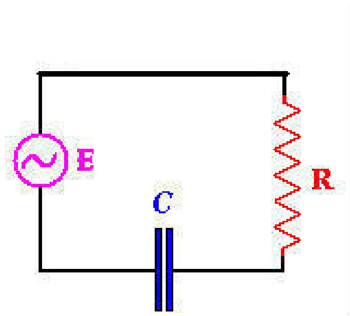
$$V_R + V_C = \mathcal{E}(t)$$



On the capacitor, the ratio of the charge to the voltage drop remains constant:

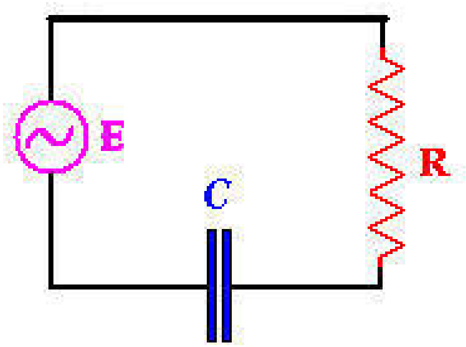
$$\frac{Q}{V_C} = C$$

$$V_C = \frac{1}{C} Q$$



Ohm's Law: The voltage drop across the resistor is proportional to the current.

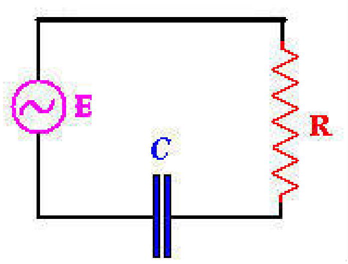
$$V_R = RI$$



Ohm's Law: The voltage drop across the resistor is proportional to the current.

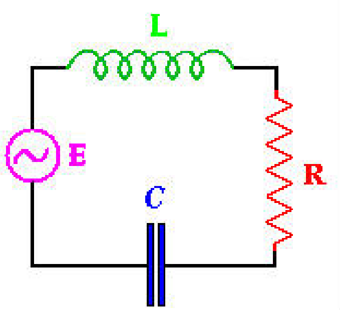
$$V_R = RI = R\frac{dQ}{dt}$$

where Q stands for the charge (in coulombs)



$$V_R + V_C = \mathcal{E}(t)$$

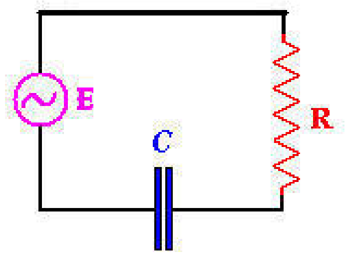
$$R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}(t)$$



RC circuit. Take $R = 2$, $C = 1$, $\mathcal{E}(t) = e^{-t}$, $Q(0) = 1$

$$R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}(t)$$

$$2\frac{dQ}{dt} + Q = e^{-t}$$



Put the equation in standard form:

$$2\frac{dQ}{dt} + Q = e^{-t}$$

$$\frac{dQ}{dt} + \frac{1}{2}Q = \frac{1}{2}e^{-t}$$

$$\mu = e^{\int \frac{1}{2}\,dt} = e^{\frac{1}{2}t}$$

$$\frac{dQ}{dt} + \frac{1}{2}Q = \frac{1}{2}e^{-t}$$

$$e^{\frac{1}{2}t}\frac{dQ}{dt} + \frac{1}{2}e^{\frac{1}{2}t}Q = \frac{1}{2}e^{-t} \cdot e^{\frac{1}{2}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{2}t}Q\right) = \frac{1}{2}e^{-\frac{1}{2}t}$$

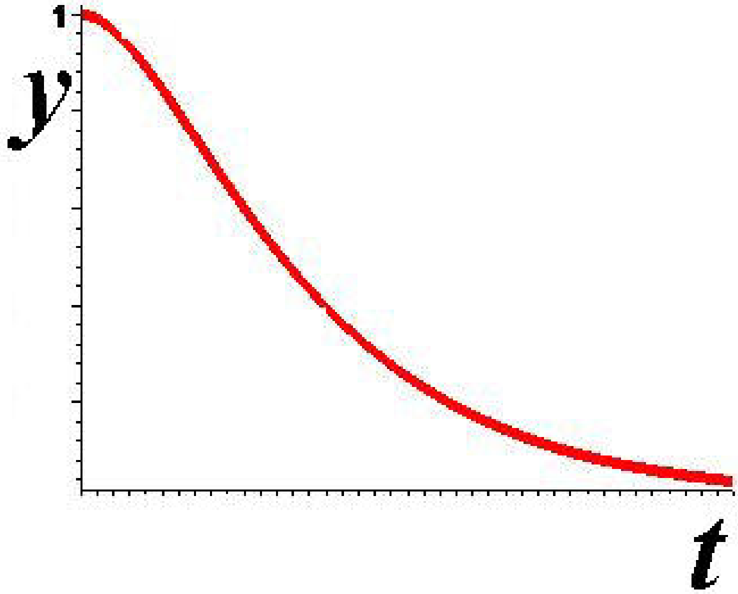
$$\frac{d}{dt}\left(e^{\frac{1}{2}t}Q\right) = \frac{1}{2}e^{-\frac{1}{2}t}$$

$$e^{\frac{1}{2}t}Q = -e^{-\frac{1}{2}t} + C$$

$$Q = -e^{-t} + Ce^{-\frac{1}{2}t}$$

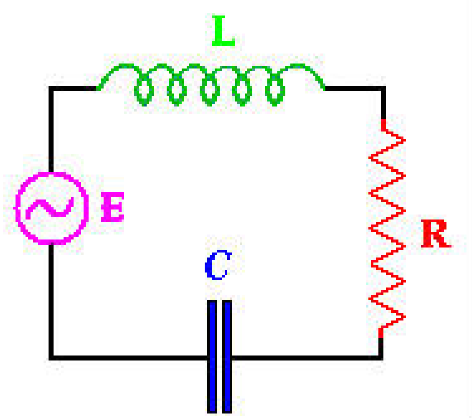
The initial condition $Q(0) = 1$ implies $C = 2$

$$Q(t) = 2e^{-t/2} - e^{-t}$$



$$V_L + V_R + V_C = \mathcal{E}(t)$$

where V_L is the voltage drop across the inductor



$$V_L = L\frac{dI}{dt}$$

where L is a constant called the *inductance* and I is the *current*

$$I = \frac{dQ}{dt}$$

$$V_L = L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$$

$$V_L + V_R + V_C = \mathcal{E}(t)$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}(t)$$

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