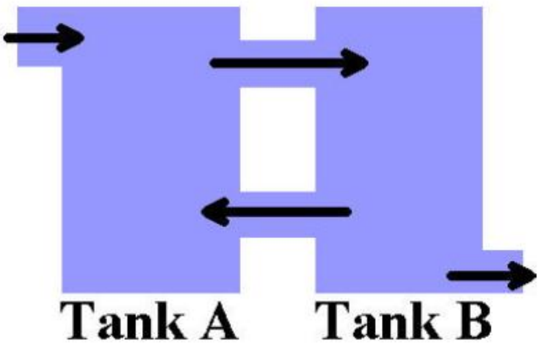
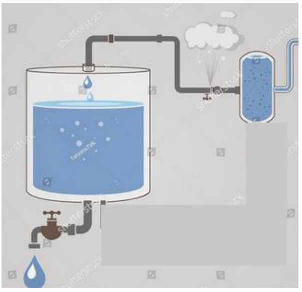
The Tank Problem
Dr. E. Jacobs

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{\mathbf{x}}$$



At $t = 0$, a 16 liter tank is filled with pure water. Then we begin pumping in a salt solution containing $\frac{1}{8}$ gram/liter of salt at the rate of 2 liters/min. At the same time, we start draining the tank at 2 liters/min.



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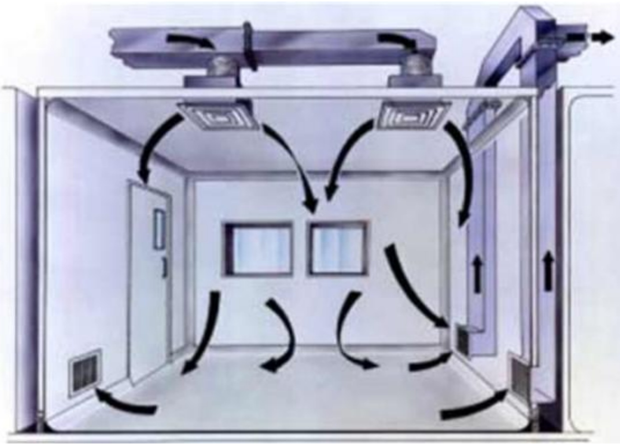
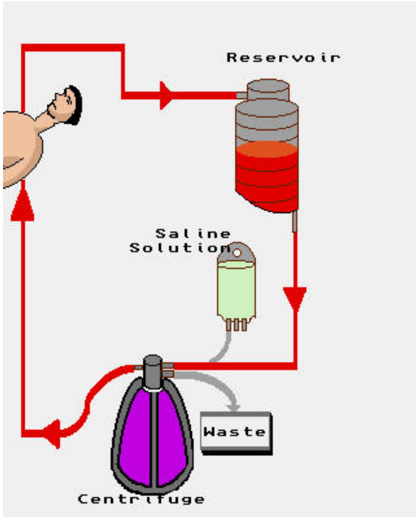
Let $x(t)$ denote the number of grams of salt in the tank after t minutes.

Let $x(t)$ be the mass of a pollutant in a room



Let $x(t)$ be the number of mg of a drug administered to a patient.



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$$\frac{dx}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix}$$

Pump in a salt solution with $\frac{1}{8}$ gm/liter of salt at the rate of 2 liters/min. Drain the tank at 2 liters/min.

$$\begin{pmatrix}\text{Rate}\\\text{In}\end{pmatrix} = \left(\frac{1}{8}\,\frac{\text{gm}}{\text{liter}}\right)\cdot\left(2\,\frac{\text{liter}}{\text{min}}\right) = \frac{1}{4}\,\frac{\text{gm}}{\text{min}}$$

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$$\begin{pmatrix}\text{Rate}\\ \text{Out}\end{pmatrix} = \left(\frac{x}{16}\,\frac{\text{gm}}{\text{liter}}\right)\cdot\left(2\,\frac{\text{liter}}{\text{min}}\right) = \frac{x}{8}\,\frac{\text{gm}}{\text{min}}$$

$$\frac{dx}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix} = \frac{1}{4} - \frac{x}{8}$$

$$\frac{dx}{dt} + \frac{1}{8}x = \frac{1}{4}$$

$$\mu = e^{\int \frac{1}{8}\, dt} = e^{\frac{1}{8}t}$$

$$\frac{dx}{dt} + \frac{1}{8}x = \frac{1}{4}$$

$$e^{\frac{1}{8}t}\frac{dx}{dt} + \frac{1}{8}e^{\frac{1}{8}t}x = \frac{1}{4}e^{\frac{1}{8}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{8}t}x\right) = \frac{1}{4}e^{\frac{1}{8}t}$$

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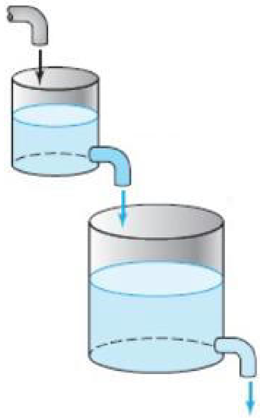
$$e^{\frac{1}{8}t}x = \int \frac{1}{4}e^{\frac{1}{8}t}\,dt = 2e^{\frac{1}{8}t} + C$$

If $x(0) = 0$ then $C = -2$

$$x(t) = 2 - 2e^{-t/8}$$

Two tanks, each 16 liters. $x(t)$ and $y(t)$ are the amounts of salt in each tank. Assume $x(0) = 1$ and $y(0) = 0$



Two tanks, each 16 liters. $x(t)$ and $y(t)$ are the amounts of salt in each tank. Assume $x(0) = 1$ and $y(0) = 0$

Pure water enters Tank 1 at 2 liter/min.

Salt solution is pumped from Tank 1 to Tank 2 at 2 liters/min

Salt solution is pumped out of Tank 2 at 2 liters/min

Find $x(t)$ and $y(t)$

Pure water enters Tank 1 at 2 liter/min.

Salt solution is pumped from Tank 1
to Tank 2 at 2 liters/min

Salt solution is pumped out of Tank 2
at 2 liters/min

$$\frac{dx}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix}$$

Pure water enters Tank 1 at 2 liter/min.

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$$\begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} = 0$$

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$$\begin{pmatrix}\text{Rate}\\ \text{In}\end{pmatrix} = 0$$

$$\begin{pmatrix}\text{Rate}\\ \text{Out}\end{pmatrix} = \left(\frac{x}{16}\,\frac{\text{gm}}{\text{liter}}\right)\cdot\left(2\,\frac{\text{liter}}{\text{min}}\right) = \frac{x}{8}\,\frac{\text{gm}}{\text{min}}$$

$$\frac{dx}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix} = 0 - \frac{x}{8}$$

Salt solution is pumped from Tank 1 to Tank 2 at 2 liters/min

Salt solution is pumped out of Tank 2 at 2 liters/min

$$\frac{dy}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix}$$

$$\begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} = \left(\frac{x}{16} \, \frac{\text{gm}}{\text{liter}} \right) \cdot \left(2 \, \frac{\text{liter}}{\text{min}} \right) = \frac{x}{8} \, \frac{\text{gm}}{\text{min}}$$

Salt solution is pumped from Tank 1 to Tank 2 at 2 liters/min

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$$\frac{dy}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix}$$

$$\begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix} = \left(\frac{y}{16}\,\frac{\text{gm}}{\text{liter}}\right) \cdot \left(2\,\frac{\text{liter}}{\text{min}}\right) = \frac{y}{8}\,\frac{\text{gm}}{\text{min}}$$

$$\frac{dy}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix} = \frac{x}{8} - \frac{y}{8}$$

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$$\text{If } \frac{dx}{dt} = -\frac{x}{8} \text{ then } x = ae^{-\frac{1}{8}t}$$

The initial condition $x(0) = 1$ implies that $a = 1$

$$x(t) = e^{-\frac{1}{8}t}$$

$$\frac{dy}{dt} = \frac{x}{8} - \frac{y}{8} = \frac{1}{8}e^{-\frac{1}{8}t} - \frac{y}{8}$$

$$\frac{dy}{dt} + \frac{1}{8}y = \frac{1}{8}e^{-\frac{1}{8}t}$$

$$\frac{dy}{dt} + \frac{1}{8}y = \frac{1}{8}e^{-\frac{1}{8}t}$$

$$\mu = e^{\int \frac{1}{8}\,dt} = e^{\frac{1}{8}t}$$

$$e^{\frac{1}{8}t}\frac{dy}{dt} + e^{\frac{1}{8}t}y = \frac{1}{8}e^{-\frac{1}{8}t} \cdot e^{\frac{1}{8}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{8}t}y\right) = \frac{1}{8}$$

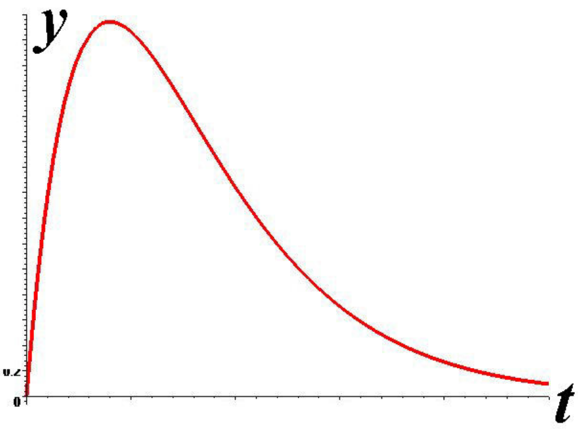
$$\frac{d}{dt}\left(e^{\frac{1}{8}t}y\right) = \frac{1}{8}$$

$$e^{\frac{1}{8}t}y = \frac{1}{8}t + C$$

The initial condition $y(0) = 0$ implies $C = 0$

$$y = \frac{1}{8}te^{-\frac{1}{8}t}$$

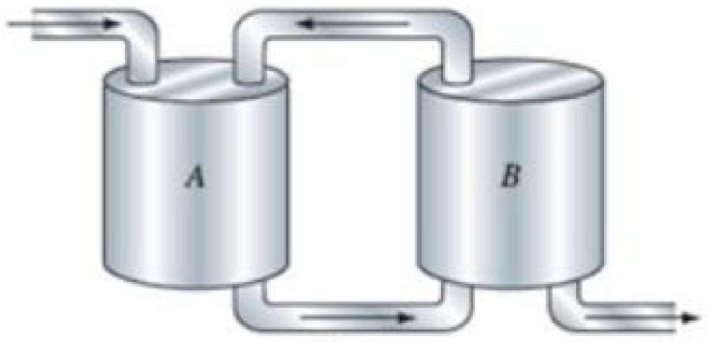
$$y = \frac{1}{8} t e^{-\frac{1}{8} t}$$



We have just solved a *system* of differential equations.

$$\begin{aligned}\frac{dx}{dt} &= -\frac{x}{8} \\ \frac{dy}{dt} &= \quad \frac{x}{8} - \frac{y}{8}\end{aligned}$$

An interconnected tank problem.



Tanks A and B have 8 liters of brine.
Pure water flowing into A at 3 liters/min
Brine flowing from A to B at 4 liters/min
Brine flowing from B to A at 1 liter/min
Brine is draining from B through another pipe at 3 liter/min.

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$$\begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} = (1\,\text{liter/min})(\frac{y}{8}\,\text{gm/liter}) = \frac{y}{8}$$

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$$\begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix} = (4\,\text{liter/min})(\frac{x}{8}\,\text{gm/liter}) = \frac{x}{2}$$

$$\frac{dx}{dt} = \begin{pmatrix} \text{Rate} \\ \text{In} \end{pmatrix} - \begin{pmatrix} \text{Rate} \\ \text{Out} \end{pmatrix} = \frac{y}{8} - \frac{x}{2}$$

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$$\frac{dx}{dt} = \frac{y}{8} - \frac{x}{2}$$

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MA 345 :
Differential Equations and Matrix Methods