Differential Equations and Matrix Methods

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Matrix Topics

Matrix Algebra
Matrix Reduction and Matrix Equations
Matrix Inverses
Eigenvectors and Eigenvalues

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Matrix Reduction and Matrix Equations
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And finally, back to differential equations

A matrix is a rectangular array of numbers

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ 2 by 2 matrices

$$\begin{bmatrix} 5 & 7 \end{bmatrix}$$ 1 by 2 matrix

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$ 2 by 1 matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 5 & 7 & 6 \\ 4 & 8 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$ 3 by 3 matrices

Alternate notation for matrices:

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{2 by 2 matrices}$$

$$\begin{pmatrix} 5 & 7 \end{pmatrix} \quad \text{1 by 2 matrix}$$

$$\begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad \text{2 by 1 matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 5 & 7 & 6 \\ 4 & 8 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{3 by 3 matrices}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{general 3 by 3 matrix}$$

Suppose a vector $\vec{\mathbf{u}}$ has coordinates u_1, u_2 and u_3. The following are all equivalent notations for $\vec{\mathbf{u}}$.

$$u_1\vec{\mathbf{i}} + u_2\vec{\mathbf{j}} + u_3\vec{\mathbf{k}}$$

$$\langle u_1,\ u_2,\ u_3 \rangle$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Scalar Multiplication:

If $\vec{\mathbf{u}}$ is a vector and c is a scalar, the scalar multiple $c\vec{\mathbf{u}}$ is defined as follows:

$$\vec{\mathbf{u}} = u_1\vec{\mathbf{i}} + u_2\vec{\mathbf{j}} + u_3\vec{\mathbf{k}}$$

$$c\vec{\mathbf{u}} = cu_1\vec{\mathbf{i}} + cu_2\vec{\mathbf{j}} + cu_3\vec{\mathbf{k}}$$

Scalar Multiplication:

If $\vec{\mathbf{u}}$ is a vector and c is a scalar, the scalar multiple $c\vec{\mathbf{u}}$ is defined as follows:

$$c\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix}$$

In general, if $\mathbf{A}$ is a matrix and c is a scalar, we obtain the matrix $c\mathbf{A}$ by multiplying each entry of $\mathbf{A}$ by c.
Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$2\mathbf{A} = \begin{pmatrix} 2 & 4 & 8 \\ 0 & 6 & 4 \\ 2 & 2 & 2 \end{pmatrix}$$

Vector Addition

If the coordinates of $\vec{\mathbf{u}}$ are u_1, u_2 and u_3 and the coordinates of $\vec{\mathbf{v}}$ are v_1, v_2 and v_3, then we define the vector sum (resultant) to be the vector obtained by adding corresponding coordinates.

$$\vec{\mathbf{u}} = u_1\vec{\mathbf{i}} + u_2\vec{\mathbf{j}} + u_3\vec{\mathbf{k}}$$

$$\vec{\mathbf{v}} = v_1\vec{\mathbf{i}} + v_2\vec{\mathbf{j}} + v_3\vec{\mathbf{k}}$$

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = (u_1 + v_1)\vec{\mathbf{i}} + (u_2 + v_2)\vec{\mathbf{j}} + (u_3 + v_3)\vec{\mathbf{k}}$$

Vector Addition

If the coordinates of $\vec{\mathbf{u}}$ are u_1, u_2 and u_3 and the coordinates of $\vec{\mathbf{v}}$ are v_1, v_2 and v_3, then we define the vector sum (resultant) to be the vector obtained by adding corresponding coordinates.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$$

To add two matrices $\mathbf{A}$ and $\mathbf{B}$, simply add corresponding entries.

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 5 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 5 & 4 & 4 \\ 5 & 2 & 4 \\ 6 & 6 & 5 \end{pmatrix}$$

Matrix addition is not necessarily defined:

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = ???$$

An expression of the form $c_1\mathbf{A}+c_2\mathbf{B}$ is called a *linear combination* of $\mathbf{A}$ and $\mathbf{B}$.

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 5 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

Calculate $2\mathbf{A} - \mathbf{B} = 2\mathbf{A} + (-1)\mathbf{B}$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 5 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} 2\mathbf{A} - \mathbf{B} &= 2\mathbf{A} + (-1)\mathbf{B} \\ &= \begin{pmatrix} 2 & 2 & 4 \\ 4 & 0 & 6 \\ 8 & 10 & 10 \end{pmatrix} + \begin{pmatrix} -4 & -3 & -2 \\ -3 & -2 & -1 \\ -2 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 & 2 \\ 1 & -2 & 5 \\ 6 & 9 & 10 \end{pmatrix} \end{aligned}$$

How do we multiply matrices?

$$\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$y_1 = ax_1 + bx_2$$

$$y_2 = cx_1 + dx_2$$

$$\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}}$$

$$\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}}$$

$$\begin{pmatrix} ax_1 + bx_2 \\ \boxed{cx_1 + dx_2} \end{pmatrix} = \begin{pmatrix} a & b \\ \boxed{c \quad d} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} x_1 \\ x_2 \end{matrix}} \end{pmatrix}$$

System of Linear Equations

Solve for x and y

$$2x + 3y = 4$$

$$4x + 5y = 6$$

Solve for x_1 and x_2

$$2x_1 + 3x_2 = 4$$

$$4x_1 + 5x_2 = 6$$

Solve for x_1 and x_2

$$2x_1 + 3x_2 = 4$$

$$4x_1 + 5x_2 = 6$$

$$\begin{pmatrix} 2x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Solve for x_1 and x_2

$$2x_1 + 3x_2 = 4$$

$$4x_1 + 5x_2 = 6$$

$$\begin{pmatrix} 2x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

This is now of the form $\mathbf{A}\vec{\mathbf{x}} = \left(\begin{smallmatrix} 4 \\ 6 \end{smallmatrix}\right)$. Solve for $\vec{\mathbf{x}}$

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 5 \end{pmatrix} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Calculate the product $\mathbf{A}\vec{\mathbf{x}}$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 5 \end{pmatrix} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Calculate the product $\mathbf{A}\vec{\mathbf{x}}$

$$\begin{pmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 3x_3 \\ 4x_1 + 5x_2 + 5x_3 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 5 \end{pmatrix} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Calculate the product $\mathbf{A}\vec{\mathbf{x}}$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 3x_3 \\ 4x_1 + 5x_2 + 5x_3 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 3x_3 \\ 4x_1 + 5x_2 + 5x_3 \end{pmatrix}$$

$$\begin{aligned} y_1 &= x_1 + x_2 + 2x_3 \\ y_2 &= 2x_1 + 3x_3 \\ y_3 &= 4x_1 + 5x_2 + 5x_3 \end{aligned}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} y_1 &= x_1 + x_2 + 2x_3 \\ y_2 &= 2x_1 + 3x_3 \\ y_3 &= 4x_1 + 5x_2 + 5x_3 \end{aligned}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 4 & 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} y_1 &= \ x_1 + x_2 + 2x_3 \\ y_2 &= 2x_1 \qquad\ \ + 3x_3 \\ y_3 &= 4x_1 + 5x_2 + 5x_3 \end{aligned}$$

A system of linear equations can always be written as a matrix equation $\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}}$

$$\text{Let} \quad \mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \vec{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \vec{\mathbf{x}} + \vec{\mathbf{y}} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}(\vec{\mathbf{x}} + \vec{\mathbf{y}}) &= \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \\ &= \begin{pmatrix} 2(x_1 + y_1) + 3(x_2 + y_2) \\ 4(x_1 + y_1) + 5(x_2 + y_2) \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
\mathbf{A}(\vec{\mathbf{x}} + \vec{\mathbf{y}}) &= \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \\
&= \begin{pmatrix} 2(x_1 + y_1) + 3(x_2 + y_2) \\ 4(x_1 + y_1) + 5(x_2 + y_2) \end{pmatrix} \\
&= \begin{pmatrix} 2x_1 + 3x_2 + 2y_1 + 3y_2 \\ 4x_1 + 5x_2 + 4y_1 + 5y_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}(\vec{\mathbf{x}}+\vec{\mathbf{y}}) &= \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \\
&= \begin{pmatrix} 2(x_1 + y_1) + 3(x_2 + y_2) \\ 4(x_1 + y_1) + 5(x_2 + y_2) \end{pmatrix} \\
&= \begin{pmatrix} 2x_1 + 3x_2 + 2y_1 + 3y_2 \\ 4x_1 + 5x_2 + 4y_1 + 5y_2 \end{pmatrix} \\
&= \begin{pmatrix} 2x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{pmatrix} + \begin{pmatrix} 2y_1 + 3y_2 \\ 4y_1 + 5y_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}(\vec{\mathbf{x}}+\vec{\mathbf{y}}) &= \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \\
&= \begin{pmatrix} 2(x_1 + y_1) + 3(x_2 + y_2) \\ 4(x_1 + y_1) + 5(x_2 + y_2) \end{pmatrix} \\
&= \begin{pmatrix} 2x_1 + 3x_2 + 2y_1 + 3y_2 \\ 4x_1 + 5x_2 + 4y_1 + 5y_2 \end{pmatrix} \\
&= \begin{pmatrix} 2x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{pmatrix} + \begin{pmatrix} 2y_1 + 3y_2 \\ 4y_1 + 5y_2 \end{pmatrix} \\
&= \mathbf{A}\vec{\mathbf{x}} + \mathbf{A}\vec{\mathbf{y}}
\end{aligned}
$$

$$\text{Let} \quad \mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \qquad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \vec{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$c_1\vec{\mathbf{x}} + c_2\vec{\mathbf{y}} = \begin{pmatrix} c_1x_1 + c_2y_1 \\ c_1x_2 + c_2y_2 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}(c_1\vec{\mathbf{x}} + c_2\vec{\mathbf{y}}) &= \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} c_1x_1 + c_2y_1 \\ c_1x_2 + c_2y_2 \end{pmatrix} \\ &= \begin{pmatrix} 2(c_1x_1 + c_2y_1) + 3(c_1x_2 + c_2y_2) \\ 4(c_1x_1 + c_2y_1) + 5(c_1x_2 + c_2y_2) \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
\mathbf{A}(c_1\vec{\mathbf{x}} + c_2\vec{\mathbf{y}}) &= \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} c_1x_1 + c_2y_1 \\ c_1x_2 + c_2y_2 \end{pmatrix} \\
&= \begin{pmatrix} 2(c_1x_1 + c_2y_1) + 3(c_1x_2 + c_2y_2) \\ 4(c_1x_1 + c_2y_1) + 5(c_1x_2 + c_2y_2) \end{pmatrix} \\
&= \begin{pmatrix} 2c_1x_1 + 3c_1x_2 + 2c_2y_1 + 3c_2y_2 \\ 4c_1x_1 + 5c_1x_2 + 4c_2y_1 + 5c_2y_2 \end{pmatrix} \\
&= \begin{pmatrix} 2c_1x_1 + 3c_1x_2 \\ 4c_1x_1 + 5c_1x_2 \end{pmatrix} + \begin{pmatrix} 2c_2y_1 + 3c_2y_2 \\ 4c_2y_1 + 5c_2y_2 \end{pmatrix} \\
&= c_1\mathbf{A}\vec{\mathbf{x}} + c_2\mathbf{A}\vec{\mathbf{y}}
\end{aligned}
$$

More generally, let $\quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

If $\quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \vec{\mathbf{y}} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

then $\quad \mathbf{A}(c_1\vec{\mathbf{x}} + c_2\vec{\mathbf{y}}) = c_1\mathbf{A}\vec{\mathbf{x}} + c_2\mathbf{A}\vec{\mathbf{y}}$

If T is an operation that distributes over linear combinations, then T is called a *linear operation*

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

Let Dy stand for $\dfrac{dy}{dx}$

$$D(2\sin x + 3\tan x) = 2\cos x + 3\sec^2 x$$

$$\text{Let} \quad Dy \quad \text{stand for} \quad \frac{dy}{dx}$$

$$D(2\sin x + 3\tan x) = 2D(\sin x) + 3D(\tan x)$$

If $f(x)$ and $g(x)$ are functions then if a and b are constants then:

$$D(af(x) + b(g(x)) = aD(f(x)) + bD(g(x))$$

If $f(x)$ and $g(x)$ are functions then if a and b are constants then:

$$D^2(af(x) + b(g(x)) = aD^2(f(x)) + bD^2(g(x))$$

The Derivative Operator $D^2 + aD + b$

Definition:

$$(D^2 + aD + b)f(x) = D^2 f(x) + aDf(x) + bf(x)$$

Examples. Suppose $y = f(x)$

$$(D^2 + 4)y = \frac{d^2y}{dx^2} + 4y$$

$$(D^2 - 2D)y = \frac{d^2y}{dx^2} - 2\frac{dy}{dx}$$

$$(D^2 + 6D + 2)y = \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 2y$$

$$(D^2+4)y = \frac{d^2y}{dx^2} + 4y$$

Suppose $y = af(x) + bg(x)$

$$\begin{aligned}(D^2+4)y &= (D^2+4)(af(x)+bg(x)) \\ &= D^2(af(x)+bg(x)) + 4(af(x)+bg(x))\end{aligned}$$

$$(D^2+4)y = \frac{d^2y}{dx^2} + 4y$$

Suppose $y = af(x) + bg(x)$

$$\begin{aligned}(D^2+4)y &= (D^2+4)(af(x)+bg(x)) \\ &= D^2(af(x)+bg(x)) + 4(af(x)+bg(x)) \\ &= aD^2f(x) + bD^2g(x) + 4af(x) + 4bg(x)\end{aligned}$$

$$(D^2+4)y = \frac{d^2y}{dx^2} + 4y$$

Suppose $y = af(x) + bg(x)$

$$\begin{aligned}
(D^2+4)y &= (D^2+4)(af(x)+bg(x)) \\
&= D^2(af(x)+bg(x)) + 4(af(x)+bg(x)) \\
&= aD^2f(x) + bD^2g(x) + 4af(x) + 4bg(x) \\
&= a(D^2f(x)+4f(x)) + b(D^2g(x)+4g(x)) \\
&= a(D^2+4)f(x) + b(D^2+4)g(x)
\end{aligned}$$

The following differential operator is linear:

$$L = D^2 + aD + b$$

$$L(c_1 f(x) + c_2 g(x)) = c_1 L f(x) + c_2 L g(x)$$

Differential Equation:
Find a function $y = y(x)$ that solves the equation:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

$$\left(D^2 + aD + b\right) y = 0$$

Matrix Equation:
Find a vector $\vec{\mathbf{x}}$ that solves the equation:

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$$