Differential Equations and Matrix Methods
Dr. E. Jacobs

Today's Topic : Matrix Reduction

Differential Equation:
Find a function $y = y(x)$ that solves the equation:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

$$\left(D^2 + aD + b\right) y = 0$$

Matrix Equation:
Find a vector $\vec{\mathbf{X}}$ that solves the equation:

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$$

Differential Equation:
Find a function $y = y(x)$ that solves the equation:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$

$$\left(D^2 + aD + b\right) y = f(x)$$

Matrix Equation:
Find a vector $\vec{\mathbf{X}}$ that solves the equation:

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{F}}$$

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$ax + by = f_1$$

$$cx + dy = f_2$$

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$ax + by = f_1$$

$$cx + dy = f_2$$

$$\begin{pmatrix} a & b & | & f_1 \\ c & d & | & f_2 \end{pmatrix}$$

The notation $\left(\begin{array}{cc|c} a & b & f_1 \\ c & d & f_2 \end{array}\right)$

represents the system of equations

$$\begin{aligned} ax + by &= f_1 \\ cx + dy &= f_2 \end{aligned}$$

Operations With Systems of Equations

1. Multiply both sides of an equation by a non-zero constant.

$$\begin{array}{rcrcl} -2x & + & 3y & = & 1 \\ x & + & y & = & 2 \end{array} \qquad \left(\begin{array}{rr|r} -2 & 3 & 1 \\ 1 & 1 & 2 \end{array}\right)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\begin{array}{rcrcl} -2x & + & 3y & = & 1 \\ 2x & + & 2y & = & 4 \end{array} \qquad \left(\begin{array}{rr|r} -2 & 3 & 1 \\ 2 & 2 & 4 \end{array}\right)$$

2. Interchange two equations

$$\begin{array}{rcrcr} -2x & + & 3y & = & 1 \\ 2x & + & 2y & = & 4 \end{array} \qquad \left(\begin{array}{rr|r} -2 & 3 & 1 \\ 2 & 2 & 4 \end{array}\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{array}{rcrcr} 2x & + & 2y & = & 4 \\ -2x & + & 3y & = & 1 \end{array} \qquad \left(\begin{array}{rr|r} 2 & 2 & 4 \\ -2 & 3 & 1 \end{array}\right)$$

3. Add a multiple of one equation to another

$$\begin{array}{rcrcl} 2x & + & 2y & = & 4 \\ -2x & + & 3y & = & 1 \end{array} \qquad \left(\begin{array}{rr|r} 2 & 2 & 4 \\ -2 & 3 & 1 \end{array}\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{array}{rcrcl} 2x & + & 2y & = & 4 \\ 0x & + & 5y & = & 5 \end{array} \qquad \left(\begin{array}{rr|r} 2 & 2 & 4 \\ 0 & 5 & 5 \end{array}\right)$$

Elementary Row Operations

1. Multiply a row through by a non-zero constant
2. Interchange two rows
3. Add a multiple of one row to another.

Matrix	System of Equations
$\left(\begin{array}{rr|r} -2 & 3 & 1 \\ 1 & 1 & 2 \end{array}\right)$	$\begin{array}{rcrcr} -2x & + & 3y & = & 1 \\ x & + & y & = & 2 \end{array}$

$$\Big\downarrow \text{ Replace } R_2 \text{ with } 2R_2$$

$\left(\begin{array}{rr|r} -2 & 3 & 1 \\ 2 & 2 & 4 \end{array}\right)$	$\begin{array}{rcrcr} -2x & + & 3y & = & 1 \\ 2x & + & 2y & = & 4 \end{array}$

$$\Big\downarrow \text{ Interchange } R_1 \text{ and } R_2$$

$\left(\begin{array}{rr|r} 2 & 2 & 4 \\ -2 & 3 & 1 \end{array}\right)$	$\begin{array}{rcrcr} 2x & + & 2y & = & 4 \\ -2x & + & 3y & = & 1 \end{array}$

$$\Big\downarrow \text{ Replace } R_2 \text{ with } R_1 + R_2$$

$\left(\begin{array}{rr|r} 2 & 2 & 4 \\ 0 & 5 & 5 \end{array}\right)$	$\begin{array}{rcrcr} 2x & + & 2y & = & 4 \\ 0x & + & 5y & = & 5 \end{array}$

$$\left(\begin{array}{cc|c} 2 & 2 & 4 \\ 0 & 5 & 5 \end{array}\right) \qquad \begin{array}{rcrcr} 2x & + & 2y & = & 4 \\ 0x & + & 5y & = & 5 \end{array}$$

$$\Big\downarrow \text{Replace } R_1 \text{ with } \tfrac{1}{2}R_1 \text{ and}$$

$$\Big\downarrow \text{Replace } R_2 \text{ with } \tfrac{1}{5}R_2$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{rcrcr} x & + & y & = & 2 \\ 0x & + & 1y & = & 1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{rcrcl} x & + & y & = & 2 \\ 0x & + & 1y & = & 1 \end{array}$$

$$\Big\downarrow \text{Replace } R_1 \text{ with } R_1 - R_2$$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{rcrcl} 1x & + & 0y & = & 1 \\ 0x & + & 1y & = & 1 \end{array}$$

The final matrix is in *Reduced Row Echelon Form*
The process demonstrated above is *Matrix Reduction*
The use of matrix reduction to reduce a matrix to reduced row echelon form is *Gauss-Jordan Elimination*

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$$

$$\text{Solve for} \quad \vec{\mathbf{X}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$$

$$\begin{aligned} 1x + 0y + 1z &= 3 \\ 2x + 2y + 4z &= 8 \\ 1x + 1y + 3z &= 5 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 2 & 2 & 4 & 8 \\ 1 & 1 & 3 & 5 \end{array}\right)$$

Suppose the solution is $x = a$, $y = b$ and $z = c$

$$\begin{aligned} 1x + 0y + 0z &= a \\ 0x + 1y + 0z &= b \\ 0x + 0y + 1z &= c \end{aligned}$$

As an augmented matrix this would be:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

This is the goal

Current matrix: $$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 2 & 2 & 4 & 8 \\ 1 & 1 & 3 & 5 \end{array}\right)$$

Goal: $$\left(\begin{array}{ccc|c} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{array}\right)$$

Current matrix: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 2 & 2 & 4 & 8 \\ 1 & 1 & 3 & 5 \end{array}\right)$

Row operation: $-2R_1 + R_2 \longrightarrow R_2$

Result: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \end{array}\right)$

Current matrix: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \end{array}\right)$

Row operation: $-R_1 + R_3 \longrightarrow R_3$

Result: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right)$

Current matrix: $$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

Goal: $$\left(\begin{array}{ccc|c} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{array}\right)$$

Current matrix: $$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

Row operation: $\frac{1}{2}R_2 \longrightarrow R_2$

Result: $$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

Current matrix: $$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

Row operation: $-R_2 + R_3 \longrightarrow R_3$

Result: $$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Progress check

Current matrix: $$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Goal: $$\left(\begin{array}{ccc|c} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{array}\right)$$

Current matrix: $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$

Row operations: $-R_3 + R_2 \longrightarrow R_2$

$-R_3 + R_1 \longrightarrow R_1$

Result: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

This is reduced row echelon form. The solution to our equation is:

$$\vec{\mathbf{X}} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

You should always check your answer:

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

You should always check your answer:

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$$

The answer checks