Differential Equations and Matrix Methods
Dr. E. Jacobs

Today's Topic : Matrix Equations

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{F}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$ax + by = f_1$$

$$cx + dy = f_2$$

The notation $\left(\begin{array}{cc|c} a & b & f_1 \\ c & d & f_2 \end{array}\right)$

represents the system of equations

$$\begin{aligned} ax + by &= f_1 \\ cx + dy &= f_2 \end{aligned}$$

Elementary Row Operations

1. Multiply a row through by a non-zero constant
2. Interchange two rows
3. Add a multiple of one row to another.

$$\text{Solve for} \quad \vec{\mathbf{X}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Suppose the solution is $x = a$, $y = b$ and $z = c$

$$\begin{aligned} 1x + 0y + 0z &= a \\ 0x + 1y + 0z &= b \\ 0x + 0y + 1z &= c \end{aligned}$$

As an augmented matrix this would be:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Set up the augmented matrix and start reducing

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

Current matrix: $$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

Row operation: $-R_2 + R_3 \longrightarrow R_3$

Result: $$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Current matrix: $$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Row operation: $-R_1 + R_2 \longrightarrow R_2$

Result: $$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Current matrix: $$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Row operation: $\frac{1}{3}R_2 \longrightarrow R_2$

Result: $$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Current matrix: $\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

Row operation: $2R_2 + R_1 \longrightarrow R_1$

Result: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

Row Echelon Form:
The first nonzero number in each row is a **1**
The leading 1's move right as you move down.

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Row Echelon Form:
Only **0**'s above and below each leading **1**

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The last row means that:

$$0x + 0y + 0z = 1$$

There is **no solution**

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

The second row means that

$$1x + 1y + 1z = 0$$

The third row means that

$$1x + 1y + 1z = 1$$

These are **inconsistent**

Example: Solve the following matrix equation:

$$\begin{pmatrix} 1 & -1 & -2 \\ -1 & 0 & 2 \\ 2 & -2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write as an augmented matrix and reduce

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & 0 & 2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right)$$

$$\text{Current matrix:} \qquad \left(\begin{array}{rrr|r} 1 & -1 & -2 & 0 \\ -1 & 0 & 2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right)$$

Row operation: $R_1 + R_2 \longrightarrow R_2$

Also: $-2R_1 + R_3 \longrightarrow R_3$

$$\text{Result:} \qquad \left(\begin{array}{rrr|r} 1 & -1 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{Current matrix:} \qquad \left(\begin{array}{rrr|r} 1 & -1 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Row operation: $-R_2 \longrightarrow R_2$

$$\text{Result:} \qquad \left(\begin{array}{rrr|r} 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{Current matrix:} \quad \left(\begin{array}{rrr|r} 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Row operation: $R_1 + R_2 \longrightarrow R_1$

$$\text{Result:} \quad \left(\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This means that:

$$\begin{aligned} 1x + 0y - 2z &= 0 \\ 0x + 1y + 0z &= 0 \\ 0x + 0y + 0z &= 0 \end{aligned}$$

$$\text{So,} \quad x = 2z \quad \text{and} \quad y = 0$$

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$$\vec{\mathbf{X}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ 0 \\ z \end{pmatrix} = z \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

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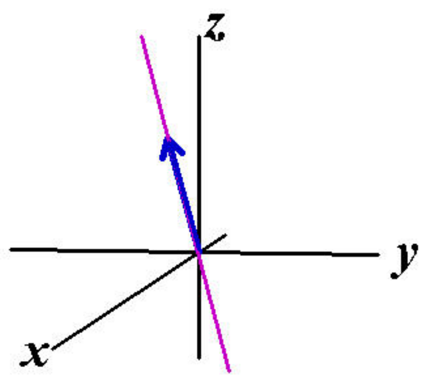

Example: Solve the following matrix equation:

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Write as an augmented matrix and reduce

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & 1 & 2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right)$$

Current matrix: $$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & 1 & 2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right)$$

Perform the usual row reduction operations

Result: $$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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The second and third row say that $0x + 0y + 0z = 0$ but the first row says that:

$$1x - y - 2z = 0$$

Result:

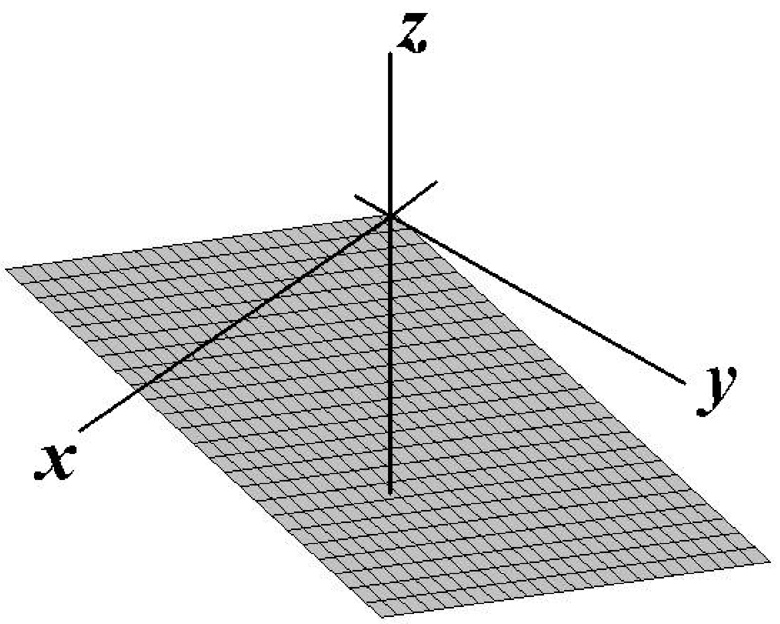
$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second and third row say that $0x + 0y + 0z = 0$ but the first row says that:

$$1x - y - 2z = 0$$

$$ax + by + cz = d$$

$$1x - y - 2z = 0$$



$$1x - y - 2z = 0$$

$$x = y + 2z$$

$$\vec{\mathbf{X}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y + 2z \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 2z \\ 0 \\ z \end{pmatrix}$$

$$1x - y - 2z = 0$$

$$x = y + 2z$$

$$\vec{\mathbf{X}} == y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

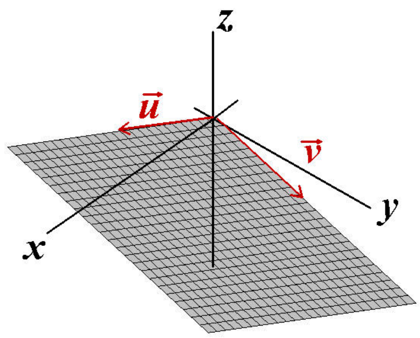
$$\vec{\mathbf{X}} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Let } \vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and let } \vec{\mathbf{v}} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

The solution $\vec{\mathbf{X}}$ is a *linear combination* of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$

$$\vec{\mathbf{X}} = y\vec{\mathbf{u}} + z\vec{\mathbf{v}}$$

$\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are the **basis** of the solution space



$$(D^2 + 4)y = 0$$

$$(D^2 - D)y = 0$$

$$(D^2 + 2D + 1)y = 0$$