Matrix Multiplication

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$$\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}}$$

$$\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}}$$

$$\begin{pmatrix} ax_1 + bx_2 \\ \boxed{cx_1 + dx_2} \end{pmatrix} = \begin{pmatrix} a & b \\ \boxed{c \quad d} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} x_1 \\ x_2 \end{matrix}} \end{pmatrix}$$

$$
\begin{array}{c}
\qquad\qquad\qquad j^{th} column \\
i^{th} row \begin{pmatrix} & & \vdots & & \\ & & \vdots & & \\ \cdots & & \sum_{k=1}^{n} a_{ik} b_{kj} & & \cdots \\ & & \vdots & & \\ & & \vdots & & \end{pmatrix} =
\end{array}
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1q} \\
b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2q} \\
\vdots & \vdots & & \vdots & & \vdots \\
b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{nq}
\end{pmatrix}
$$

Multiply the matrices:

$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$$

Multiply the matrices:

$$\begin{pmatrix} -7 & 12 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$$

Row 1 Column 1 of product **AB**

A **B**

$$\begin{pmatrix} \boxed{-7} & 12 \\ -1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$$

Row 1 Column 2 of product $\mathbf{AB}$

$$\begin{pmatrix} -7 & \boxed{12} \\ -1 & 6 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}}_{\mathbf{B}}$$

Row 2 Column 1 of product **AB**

$$\begin{pmatrix} -7 & 12 \\ \boxed{-1} & 6 \end{pmatrix} = \overset{\mathbf{A}}{\begin{pmatrix} 1 & 4 \\ \boxed{3 \quad 2} \end{pmatrix}} \overset{\mathbf{B}}{\begin{pmatrix} \boxed{\begin{matrix} 1 \\ -2 \end{matrix}} & \begin{matrix} 0 \\ 3 \end{matrix} \end{pmatrix}}$$

Row 2 Column 2 of product **AB**

$$\begin{pmatrix} -7 & 12 \\ -1 & \boxed{6} \end{pmatrix} = \overset{\mathbf{A}}{\begin{pmatrix} 1 & 4 \\ \boxed{3 \quad 2} \end{pmatrix}} \overset{\mathbf{B}}{\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 12 \\ -1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 7 & -2 \end{pmatrix}$$

In general, $\mathbf{AB}$ is not the same as $\mathbf{BA}$

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & 0 \\ 0 & 2 \\ 1 & 3 \end{pmatrix}$$

Row1 Column 1

$$\begin{pmatrix} \boxed{1 \quad 0 \quad 1 \quad 2} \\ 0 \quad 0 \quad 1 \quad -1 \\ -1 \quad 2 \quad 3 \quad 0 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 3 \\ 4 \\ 0 \\ 1 \end{matrix}} & \begin{matrix} 6 \\ 0 \\ 2 \\ 3 \end{matrix} \end{pmatrix} = \begin{pmatrix} \boxed{5} & \\ & \\ & \end{pmatrix}$$

Row1 Column 2

$$\begin{pmatrix} \boxed{1 & 0 & 1 & 2} \\ 0 & 0 & 1 & -1 \\ -1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & 0 \\ 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & \boxed{14} & \\ & & \\ & & \end{pmatrix}$$

Row1 Column 3

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & 0 \\ 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 14 & \textbf{??} \\ & & \\ & & \end{pmatrix}$$

$$\mathbf{A} \qquad\qquad \mathbf{B} \qquad\qquad \mathbf{AB}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & 0 \\ 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 14 \\ -1 & -1 \\ 5 & 0 \end{pmatrix}$$

A **B** **AB**

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 4 & 0 \\ 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 14 \\ -1 & -1 \\ 5 & 0 \end{pmatrix}$$

1. The number of columns of **A** must equal the number of rows of **B**.

2. The number of rows of **AB** must equal the number of rows of **A**.

3. The number of columns of **AB** will always be equal to the number of columns of **B**.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

An n by n identity matrix is a matrix of the form:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

An identity matrix **I** has the property that

$$\mathbf{IA} = \mathbf{A} \;\text{ and }\; \mathbf{AI} = \mathbf{A}$$

for any n by n matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$$

Multiply these two matrices.

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Definition

If $\mathbf{A}$ and $\mathbf{B}$ are matrices with the property that

$$\mathbf{AB} = \mathbf{I}$$

and

$$\mathbf{BA} = \mathbf{I}$$

then $\mathbf{A}$ and $\mathbf{B}$ are *inverses*.

$$\mathbf{B} = \mathbf{A}^{-1}$$

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 3 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

Multiply $\mathbf{AB}$

$$\begin{array}{ccc} \mathbf{A} & \mathbf{B} & \mathbf{I} \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, $\mathbf{BA} = \mathbf{I}$.

$$\mathbf{B} = \mathbf{A}^{-1}$$

Solve for $\binom{x}{y}$

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solve for $\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solve for $\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example: Solve the following equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{F}}$$

$$\mathbf{A}^{-1}\mathbf{A}\vec{\mathbf{X}} = \mathbf{A}^{-1}\vec{\mathbf{F}}$$

$$\mathbf{I}\vec{\mathbf{X}} = \mathbf{A}^{-1}\vec{\mathbf{F}}$$

$$\vec{\mathbf{X}} = \mathbf{A}^{-1}\vec{\mathbf{F}}$$

Example: Solve the following equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 \\ -3 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$