Differential Equations and Matrix Methods

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Today's Topic : Matrix Inverse

Ultimate Goal:
Solve higher order linear differential equations

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = f(t)$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}(t)$$

We are starting with matrix equations:

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{F}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{F}}$$

$$\vec{\mathbf{X}} = \mathbf{A}^{-1}\vec{\mathbf{F}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \text{First column of } \mathbf{A}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} = \text{Second column of } \mathbf{A}$$

Given $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Let $\mathbf{B} = \mathbf{A}^{-1} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Find $\mathbf{B}$.

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{First column of } \mathbf{A}^{-1}$$

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$$\text{Let } \vec{\mathbf{X}} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{First column of } \mathbf{A}^{-1}$$

$$\text{Let } \vec{\mathbf{X}} = \mathbf{A}^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \text{First column of } \mathbf{A}^{-1}$$

$$\vec{\mathbf{X}} = \mathbf{A}^{-1}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\mathbf{A}\vec{\mathbf{X}} = \mathbf{A}\mathbf{A}^{-1}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\mathbf{A}\vec{\mathbf{X}} = \begin{pmatrix}1\\0\end{pmatrix}$$

If we can solve:

$$\mathbf{A}\vec{\mathbf{X}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then we will get the first column of the inverse.
By the same reasoning, if we can solve:

$$\mathbf{A}\vec{\mathbf{X}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then we will get the second column of the inverse.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

Find the inverse matrix.

Start with the first column of the inverse

$$\mathbf{A}\vec{\mathbf{X}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Reduce the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 0 \end{array}\right)$$

Row operation: $-2R_1 + R_2 \longrightarrow R_2$

$$\left(\begin{array}{cc|r} 1 & 1 & 1 \\ 0 & 1 & -2 \end{array}\right)$$

Next row operation: $-R_2 + R_1 \longrightarrow R_1$

$$\left(\begin{array}{cc|r} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right)$$

We now have the first column of the inverse:

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & ? \\ -2 & ? \end{pmatrix}$$

To get the second column of the inverse, solve:

$$\mathbf{A}\vec{\mathbf{X}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 1 \end{array}\right)$$

Row operation: $-2R_1 + R_2 \longrightarrow R_2$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right)$$

Next row operation: $-R_2 + R_1 \longrightarrow R_1$

$$\left(\begin{array}{cc|r} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

We now have the second column of the inverse:

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$$

Check:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Whether we were solving $\mathbf{A}\vec{\mathbf{X}} = \binom{1}{0}$ or $\mathbf{A}\vec{\mathbf{X}} = \binom{0}{1}$ we used the same sequence of elementary row operations:

$$-2R_1 + R_2 \longrightarrow R_2$$

$$-R_2 + R_1 \longrightarrow R_1$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)$$

If we ignore the last column, we're solving $\mathbf{A}\vec{\mathbf{X}} = \binom{1}{0}$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)$$

If we ignore the 3rd column, we're solving $\mathbf{A}\vec{\mathbf{X}} = \binom{0}{1}$

$$\left(\begin{array}{cc|cc} 1 & 1 & \cancel{1} & 0 \\ 2 & 3 & \cancel{0} & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)$$

$$-2R_1 + R_2 \longrightarrow R_2$$

$$\left(\begin{array}{cc|rr} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

$$-R_2 + R_1 \longrightarrow R_1$$

$$\left(\begin{array}{cc|rr} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

Algorithm for finding the inverse of a matrix

$$(\mathbf{A} \mid \mathbf{I}) \xrightarrow{\text{reduce}} (\mathbf{I} \mid \mathbf{A}^{-1})$$

Find the inverse

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Augment with the identity matrix and then reduce

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Apply $-R_2 + R_1 \longrightarrow R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Apply $-R_3 + R_2 \longrightarrow R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the inverse

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$-R_1 + R_2 \longrightarrow R_2$$
$$-R_1 + R_3 \longrightarrow R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right)$$

How can we tell if a given matrix has an inverse?