Differential Equations and Matrix Methods
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Today's Topic: Linear Independence and Invertibility

Algorithm for finding the inverse of a matrix

$$(\mathbf{A} \mid \mathbf{I}) \xrightarrow{\text{reduce}} (\mathbf{I} \mid \mathbf{A}^{-1})$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

$$
\begin{aligned}
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} \\
&= \begin{pmatrix} a_{11}x_1 \\ a_{21}x_1 \end{pmatrix} + \begin{pmatrix} a_{12}x_2 \\ a_{22}x_2 \end{pmatrix}
\end{aligned}
$$

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&= \begin{pmatrix} a_{11}x_1 \\ a_{21}x_1 \end{pmatrix} + \begin{pmatrix} a_{12}x_2 \\ a_{22}x_2 \end{pmatrix} \\
&= x_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}
\end{aligned}
$$

Solve for $\vec{\mathbf{X}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ so that $\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Equivalently, find numbers x_1 and x_2 so that:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equivalently, find numbers x_1, x_2 and x_3 so that:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Find numbers x_1, x_2 and x_3 so that:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

One possibility is $x_1 = x_2 = x_3 = 0$

$$0 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + 0 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + 0 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Find numbers x_1, x_2 and x_3 so that:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If at least some of the x's are nonzero, then this is called a nontrivial solution.

If there is a set of numbers x_1, x_2, …, x_n that are not all zero (a nontrivial set) so that

$$x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + \cdots + x_n\vec{\mathbf{v}}_n = \vec{\mathbf{0}}$$

then the set of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ is said to be linearly dependent.

Example:

$$\text{Let } \vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \vec{\mathbf{v}}_2 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \qquad \vec{\mathbf{v}}_3 = \begin{pmatrix} 9 \\ 6 \\ 7 \end{pmatrix}$$

$$3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} 9 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3\vec{\mathbf{v}}_1 + 2\vec{\mathbf{v}}_2 + (-1)\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$$

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This is equivalent to:

$$\vec{\mathbf{v}}_3 = 3\vec{\mathbf{v}}_1 + 2\vec{\mathbf{v}}_2$$

If a set of vectors is linearly dependent then at least one of the vectors must be a linear combination of the other vectors.

Example: Find the vector $\vec{\mathbf{X}}$ that solves the following equation:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that $\vec{\mathbf{X}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ always solves $\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $\mathbf{A}$ has an inverse, then $\vec{\mathbf{X}} = \vec{\mathbf{0}}$ is the *only* solution of $\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$.

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$$

$$\mathbf{A}^{-1}\mathbf{A}\vec{\mathbf{X}} = \mathbf{A}^{-1}\vec{\mathbf{0}}$$

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$$\mathbf{I}\vec{\mathbf{X}} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{X}} = \vec{\mathbf{0}}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $\mathbf{A}$ has an inverse, then $\vec{\mathbf{X}} = \vec{\mathbf{0}}$ is the *only* solution of $\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$.

This implies that if there is a nontrivial solution ($\vec{\mathbf{X}} \neq \vec{\mathbf{0}}$) then $\mathbf{A}$ will not have an inverse.

Example: Find the vector $\vec{\mathbf{X}}$ that solves the following equation:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example: Find the vector $\vec{\mathbf{X}}$ that solves the following equation:

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$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 \vec{\mathbf{v}}_1 + x_2 \vec{\mathbf{v}}_2 + x_3 (2\vec{\mathbf{v}}_1 + 3\vec{\mathbf{v}}_2) = \vec{\mathbf{0}}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + x_3(2\vec{\mathbf{v}}_1 + 3\vec{\mathbf{v}}_2) = \vec{\mathbf{0}}$$

$$x_1 = -2x_3 \qquad\qquad x_2 = -3x_3$$

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Let $t = x_3$

$$x_1 = -2t \qquad\qquad x_2 = -3t$$

$$\vec{\mathbf{X}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2t \\ -3t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$$

Since $\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$ has nontrivial solutions, $\mathbf{A}$ couldn't possibly have an inverse.

A matrix **A** will not have an inverse if the columns of the matrix form a linearly dependent set of vectors.

In this case, **A** is said to be a *singular matrix*.