

Differential Equations and Matrix Methods
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Today's Topic : Determinants

$$ax + by = f_1$$

$$cx + dy = f_2$$

Solution:

$$x = \frac{f_1 d - f_2 b}{ad - bc} \qquad y = \frac{f_2 a - f_1 c}{ad - bc}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Alternative notation. If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then:

$$\det(\mathbf{A}) = ad - bc$$

$$ax + by = f_1$$

$$cx + dy = f_2$$

Solution:

$$x = \frac{f_1 d - f_2 b}{ad - bc} = \frac{\begin{vmatrix} f_1 & b \\ f_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$ax + by = f_1$$

$$cx + dy = f_2$$

Solution:

$$y = \frac{f_2a - f_1c}{ad - bc} = \frac{\begin{vmatrix} a & f_1 \\ c & f_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let M_{ij} denote the determinant obtained if the i^{th} row and j^{th} column is deleted. This is called the Minor of the i^{th} row, j^{th} column.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$M_{11} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$M_{11} = a_{22}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$M_{12} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$M_{12} = a_{21}$$

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= a_{11}a_{22} - a_{12}a_{21} \\ &= a_{11}M_{11} - a_{12}M_{12} \end{aligned}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{11} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = ?$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Calculate $\begin{vmatrix} 4 & 3 & 2 \\ 1 & 3 & 0 \\ 2 & 2 & 1 \end{vmatrix}$

Calculate $\begin{vmatrix} 4 & 3 & 2 \\ 1 & 3 & 0 \\ 2 & 2 & 1 \end{vmatrix}$

$$= 4 \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}$$

$$= 4(3 - 0) - 3(1 - 0) + 2(2 - 6) = 12 - 3 - 8 = 1$$

$$\text{Calculate } \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 2 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 0 \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} \\ &= 0(3 - 0) - 0(1 - 0) + 0(2 - 6) = 0 \end{aligned}$$

$$\vec{\mathbf{u}} = 1\vec{\mathbf{i}} + 2\vec{\mathbf{j}} + 1\vec{\mathbf{k}} \qquad \vec{\mathbf{v}} = 0\vec{\mathbf{i}} - 1\vec{\mathbf{j}} + 2\vec{\mathbf{k}}$$

Calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$

$$\begin{aligned}\vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \vec{\mathbf{k}} \\ &= 5\vec{\mathbf{i}} - 2\vec{\mathbf{j}} - \vec{\mathbf{k}}\end{aligned}$$

Algorithm for finding the inverse of a matrix

$$(\mathbf{A} \mid \mathbf{I}) \xrightarrow{\text{reduce}} (\mathbf{I} \mid \mathbf{A}^{-1})$$

Find the inverse

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

↓

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

If $\mathbf{A}$ has an inverse, it can be reduced to the identity matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the inverse

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

↓

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right)$$

If $\mathbf{A}$ has no inverse, then the reduced row echelon form is not the identity matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

How do the elementary row operations effect the value of a determinant?

1. Interchange two rows
2. Multiply a row by a nonzero constant
3. Add a multiple of one row to another.

Interchange two rows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{12}a_{21} - a_{11}a_{22}$$

Interchange two rows:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{12}a_{21} - a_{11}a_{22}$$

Interchanging two rows has the effect of changing the determinant by a minus sign

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix} = ka_{11}a_{22} - ka_{12}a_{21}$$

Multiplying a row by a non-zero constant multiplies the entire determinant by that constant.

$$\begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Add a multiple of one row to another

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} + ka_{11} & a_{22} + ka_{12} \end{vmatrix}$$

Add a multiple of one row to another

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} + ka_{11} & a_{22} + ka_{12} \end{vmatrix}$$

$$= a_{11}(a_{22} + ka_{12}) - a_{12}(a_{21} + ka_{11}) = a_{11}a_{22} - a_{12}a_{21}$$

Adding a multiple of one row to another does not change the value of the determinant

Interchanging two rows has the effect of changing the determinant by a minus sign

Multiplying a row by a non-zero constant multiplies the entire determinant by that constant.

Adding a multiple of one row to another does not change the value of the determinant

$$\mathbf{A} \longrightarrow \mathbf{B}$$

Suppose matrix $\mathbf{A}$ is transformed into matrix $\mathbf{B}$ by interchanging two rows

$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

If $\det(\mathbf{A}) = 0$ then $\det(\mathbf{B})$ will be 0 also.

If $\det(\mathbf{A})$ is not 0 then $\det(\mathbf{B})$ won't be 0 either

$$\mathbf{A} \longrightarrow \mathbf{B}$$

Suppose matrix $\mathbf{A}$ is transformed into matrix $\mathbf{B}$ by multiplying a row by a nonzero constant c

$$\det(\mathbf{B}) = c \cdot \det(\mathbf{A})$$

If $\det(\mathbf{A}) = 0$ then $\det(\mathbf{B})$ will be 0 also.

If $\det(\mathbf{A})$ is not 0 then $\det(\mathbf{B})$ won't be 0 either

$$\mathbf{A} \longrightarrow \mathbf{B}$$

Suppose matrix $\mathbf{A}$ is transformed into matrix $\mathbf{B}$ by adding a multiple of one row to another row

$$\det(\mathbf{B}) = \det(\mathbf{A})$$

If $\det(\mathbf{A}) = 0$ then $\det(\mathbf{B})$ will be 0 also.

If $\det(\mathbf{A})$ is not 0 then $\det(\mathbf{B})$ won't be 0 either

$$\mathbf{A} \longrightarrow \mathbf{B}$$

Suppose matrix $\mathbf{A}$ is transformed into matrix $\mathbf{B}$ by performing a sequence of elementary row operations

If $\det(\mathbf{A}) = 0$ then $\det(\mathbf{B})$ will be 0 also.

If $\det(\mathbf{A})$ is not 0 then $\det(\mathbf{B})$ won't be 0 either

If $\mathbf{A}^{-1}$ exists then $\mathbf{A}$ can be reduced to the identity matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This can't happen if the determinant of $\mathbf{A}$ equals 0

Theorem: If $\det(\mathbf{A}) = 0$ then $\mathbf{A}^{-1}$ doesn't exist

Example: Can we find the inverse of the following matrix?

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 3 & 3 \\ 1 & 5 & 4 \end{pmatrix}$$

$$\begin{aligned}
 \begin{vmatrix} 1 & 3 & 2 \\ 0 & 3 & 3 \\ 1 & 5 & 4 \end{vmatrix} &= 1 \begin{vmatrix} 3 & 3 \\ 5 & 4 \end{vmatrix} - 3 \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} \\
 &= 1(12 - 15) - 3(0 - 3) + 2(0 - 3) \\
 &= 0
 \end{aligned}$$

The matrix has no inverse