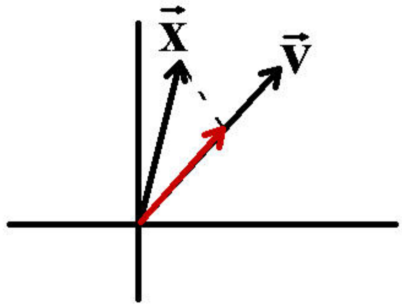
Differential Equations and Matrix Methods
Dr. E. Jacobs

Matrix Equation - Projection Example

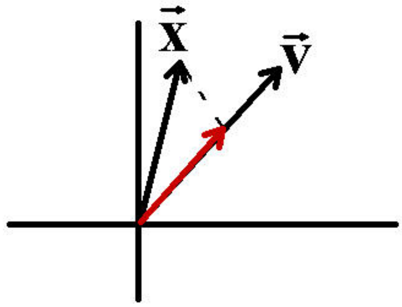
$\text{proj}_{\vec{\mathbf{v}}}\,\vec{\mathbf{x}}$ means the projection $\vec{\mathbf{x}}$ in the direction of $\vec{\mathbf{v}}$

$$\text{proj}_{\vec{\mathbf{v}}}\,\vec{\mathbf{x}} = \frac{\vec{\mathbf{x}} \bullet \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|^2}\,\vec{\mathbf{v}}$$



Let $\vec{\mathbf{x}} = x_1\vec{\mathbf{i}} + x_2\vec{\mathbf{j}}$ and let $\vec{\mathbf{v}} = 1\vec{\mathbf{i}} + 1\vec{\mathbf{j}}$

$$\text{proj}_{\vec{\mathbf{v}}}\,\vec{\mathbf{x}} = \frac{\vec{\mathbf{x}} \bullet \vec{\mathbf{v}}}{|\vec{\mathbf{v}}|^2}\,\vec{\mathbf{v}} = \frac{x_1 + x_2}{2}\vec{\mathbf{v}}$$



$$
\begin{aligned}
\operatorname{proj}_{\vec{\mathbf{v}}} \vec{\mathbf{x}} &= \frac{x_1 + x_2}{2} \vec{\mathbf{v}} \\
&= \frac{x_1 + x_2}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{proj}_{\vec{\mathbf{v}}} \vec{\mathbf{x}} &= \frac{x_1 + x_2}{2} \vec{\mathbf{v}} \\
&= \frac{x_2 + x_2}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}
$$

$$\operatorname{proj}_{\vec{\mathbf{v}}} \vec{\mathbf{x}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This is now in the form $\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}}$.

Solve the equation $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solve the equation $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This can only happen if:

$$x_1 + x_2 = 0$$

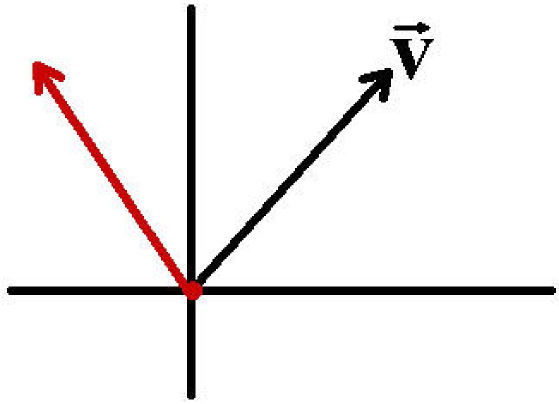
$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

Let $t = x_2$, so $x_1 = -t$

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{\mathbf{x}} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Solve for $\vec{\mathbf{x}}$:

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{x}}$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solve for $\vec{\mathbf{x}}$:

$$\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{x}}$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1/2 + x_2/2 \\ x_1/2 + x_2/2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$$x_1 + x_2 = 2x_1 \qquad x_1 + x_2 = 2x_2$$

$$-x_1 + x_2 = 0 \qquad x_1 - x_2 = 0$$

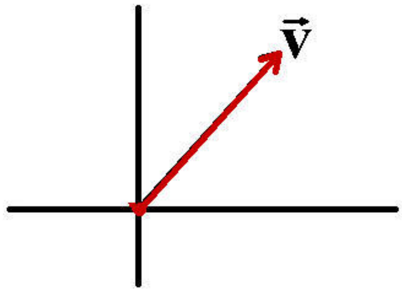
Either way, we get $x_1 = x_2$. Let $t = x_2$

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{\mathbf{x}} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = t\vec{\mathbf{v}}$$

The solution of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{x}}$ is any scalar multiple of $\vec{\mathbf{v}}$

$$\operatorname{proj}_{\vec{\mathbf{v}}} \vec{\mathbf{v}} = \vec{\mathbf{v}}$$



We have just found a nonzero solution of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{x}}$

More generally, we're going to consider equations of the form:

$$\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$$

If $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$ then $\vec{\mathbf{x}}$ is called an *eigenvector*
The scalar λ is called an *eigenvalue*.

$$\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$$

$$\mathbf{A}\vec{\mathbf{x}} - \lambda\vec{\mathbf{x}} = \vec{\mathbf{0}}$$

$$\mathbf{A}\vec{\mathbf{x}} - \lambda\mathbf{I}\vec{\mathbf{x}} = \vec{\mathbf{0}}$$

$$(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$$

$$\mathbf{A}\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$$

$$(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$$

If this equation has a nonzero solution then $\mathbf{A} - \lambda\mathbf{I}$ has no inverse

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This is called the *characteristic equation*. The solutions are the eigenvalues.

$$\mathbf{A} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Solve for λ so that $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\begin{aligned} \mathbf{A} - \lambda\mathbf{I} &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{pmatrix} \end{aligned}$$

Solve $\left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4} = 0$

Solve $\left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4} = 0$

$$\frac{1}{4} - \lambda + \lambda^2 - \frac{1}{4} = 0$$

$$-\lambda + \lambda^2 = 0$$

$$\lambda(-1 + \lambda) = 0$$

The solutions are $\lambda = 0$ and $\lambda = 1$. These are the eigenvalues.

$$\mathbf{A} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

If $\lambda = 0$ and $\lambda = 1$ are the eigenvalues then each of the following equations have nonzero solutions.

$$\mathbf{A}\vec{\mathbf{x}} = 0\vec{\mathbf{x}} \qquad \mathbf{A}\vec{\mathbf{x}} = 1\vec{\mathbf{x}}$$

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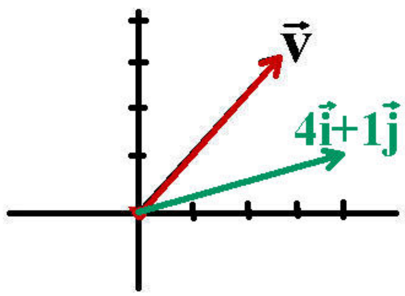
$$\mathbf{A}\vec{\mathbf{x}} = 0\vec{\mathbf{x}} \qquad \mathbf{A}\vec{\mathbf{x}} = 1\vec{\mathbf{x}}$$

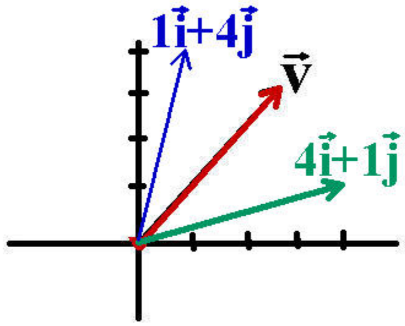
Eigenvectors:

$$\vec{\mathbf{x}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example: Reflection



Example: Reflection



$$\text{If} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{then} \quad f(\vec{\mathbf{x}}) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$\text{If} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{then} \quad f(\vec{\mathbf{x}}) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$f(\vec{\mathbf{x}}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Calculate the eigenvalues and eigenvectors of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solve the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 1$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad \text{when} \quad \lambda = \pm 1$$

Eigenvalues: $\lambda = 1$ $\lambda = -1$

$$\mathbf{A}\vec{\mathbf{x}} = 1\vec{\mathbf{x}} \qquad \mathbf{A}\vec{\mathbf{x}} = (-1)\vec{\mathbf{x}}$$

Eigenvectors:

Eigenvalues: $\lambda = 1$ $\lambda = -1$

$$\mathbf{A}\vec{\mathbf{x}} = 1\vec{\mathbf{x}} \qquad \mathbf{A}\vec{\mathbf{x}} = (-1)\vec{\mathbf{x}}$$

Eigenvectors:

Brief Intermission

Pause the video and find the eigenvectors of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

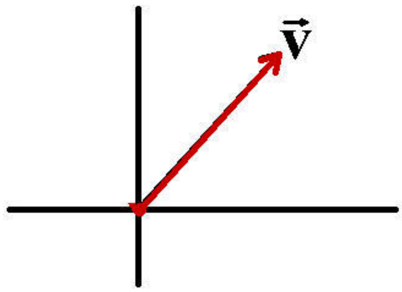
Eigenvalues: $\lambda = 1$ $\lambda = -1$

$$\mathbf{A}\vec{\mathbf{x}} = 1\vec{\mathbf{x}} \qquad \mathbf{A}\vec{\mathbf{x}} = (-1)\vec{\mathbf{x}}$$

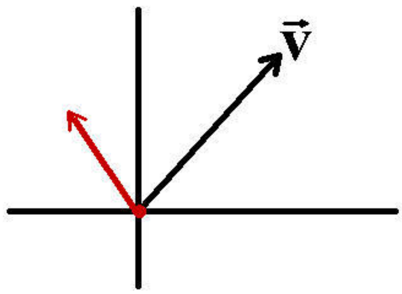
Eigenvectors:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

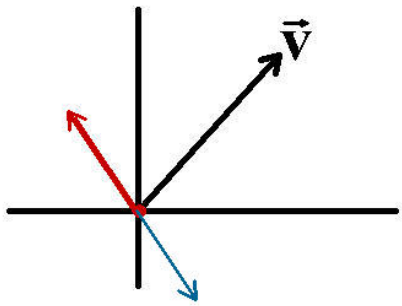
$$\lambda = 1 \qquad \mathbf{A}\vec{\mathbf{x}} = 1\vec{\mathbf{x}} \qquad \vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$\lambda = -1$ $\mathbf{A}\vec{\mathbf{x}} = (-1)\vec{\mathbf{x}}$ $\vec{\mathbf{x}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$$\lambda = -1 \qquad \mathbf{A}\vec{\mathbf{x}} = (-1)\vec{\mathbf{x}} \qquad \vec{\mathbf{x}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



What's coming in the next week?

1. More eigenvector and eigenvalue practice
2. Systems of linear differential equations

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$