Differential Equations and Matrix Methods
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Today's Topic : Eigenvectors and Eigenvalues

The following statements are equivalent:

1. $\mathbf{A}$ has no inverse
2. The columns of $\mathbf{A}$ are linearly dependent
3. The determinant of $\mathbf{A}$ is zero
4. $\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$ has a nontrivial solution $\vec{\mathbf{X}}$

$$\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

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2. The columns of $\mathbf{A} - \lambda\mathbf{I}$ are linearly dependent
3. The determinant of $\mathbf{A} - \lambda\mathbf{I}$ is zero
4. $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{X}} = \vec{\mathbf{0}}$ has a nontrivial solution $\vec{\mathbf{X}}$
5. $\mathbf{A}\vec{\mathbf{X}} = \lambda\vec{\mathbf{X}}$ has a nontrivial solution $\vec{\mathbf{X}}$

Definition:

If $\mathbf{A}\vec{\mathbf{X}} = \lambda\vec{\mathbf{X}}$ has a nontrivial solution $\vec{\mathbf{X}}$, then λ is called an *eigenvalue* and $\vec{\mathbf{X}}$ is called an *eigenvector*.

To find the eigenvalues, solve the following equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This is called the *characteristic equation*

Definition:

If $\mathbf{A}\vec{\mathbf{U}} = \lambda\vec{\mathbf{U}}$ has a nontrivial solution $\vec{\mathbf{U}}$, then λ is called an *eigenvalue* and $\vec{\mathbf{U}}$ is called an *eigenvector*.

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Example: Find the eigenvalues and eigenvectors of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

Start with the eigenvalues by solving the characteristic equation.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 9 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda)$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ when $\lambda = 1$ or $\lambda = -2$
These are the eigenvalues.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

Next, find the eigenvectors.
If $\lambda = 1$, then we are solving $\mathbf{A}\vec{\mathbf{U}} = 1\vec{\mathbf{U}}$

$$(\mathbf{A} - \mathbf{I})\vec{\mathbf{U}} = \vec{\mathbf{0}}$$

$$\begin{pmatrix} 0 & 0 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 9u_1 - 3u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$9u_1 - 3u_2 = 0$$

$$u_2 = 3u_1$$

If we let $t = u_1$ then $u_2 = 3t$

$$\vec{\mathbf{U}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} t \\ 3t \end{pmatrix} = t \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

If $\lambda = -2$ then we are solving $\mathbf{A}\vec{\mathbf{U}} = -2\vec{\mathbf{U}}$

$$(\mathbf{A} + 2\mathbf{I})\vec{\mathbf{U}} = \vec{\mathbf{0}}$$

$$\begin{pmatrix} 3 & 0 \\ 9 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 9 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3u_1 \\ 9u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This implies that $u_1 = 0$.
There is no restriction on u_2. Let $t = u_2$

$$\vec{\mathbf{U}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let's check this answer.

$$\mathbf{A}\vec{\mathbf{U}} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{A}\vec{\mathbf{U}} = -2\vec{\mathbf{U}}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

Eigenvalues:	$\lambda = 1$	$\lambda = -2$
Eigenvectors:	$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix}$$

Find the eigenvectors and eigenvalues

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix}$$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 1-\lambda & 2 & 2 \\ 0 & 3-\lambda & 0 \\ 0 & 4 & -1-\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(3-\lambda)(-1-\lambda)$$

The eigenvalues must be: $\lambda = 1 \quad \lambda = 3 \quad \lambda = -1$

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$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix}$$

Let's find the eigenvector for $\lambda = 1$

$$\mathbf{A}\vec{\mathbf{U}} = 1\vec{\mathbf{U}}$$

$$(\mathbf{A} - \mathbf{I})\vec{\mathbf{U}} = \vec{\mathbf{0}}$$

$$\begin{pmatrix} 0 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 4 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 2 & | & 0 \\ 0 & 2 & 0 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{pmatrix}$$

Apply the elementary row operations and reduce:

$$\begin{pmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The first two rows imply that $u_2 = 0$ and $u_3 = 0$
No restriction on u_1. Let $t = u_1$.

$$\vec{\mathbf{U}} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix}$$

Eigenvalues :	$\lambda = 1$	$\lambda = 3$	$\lambda = -1$
Eigenvectors :	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$	$\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix}$$

$$\text{Eigenvalues :} \qquad \lambda = 1 \qquad \lambda = 3 \qquad \lambda = -1$$

$$\text{Eigenvectors :} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix} \quad \text{had eigenvectors:} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \qquad \mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$
\begin{aligned}
\mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}
\end{aligned}
$$

Diagonal Matrices

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$