# Differential Equations and Matrix Methods
**Dr. E. Jacobs**

## Today's Topic : Diagonal Matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

Multiply $\quad \mathbf{A} \cdot \mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -9 & 4 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

Calculate $\mathbf{A}^3$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -9 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 27 & -8 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix}$$

Calculate $\quad \mathbf{A}^{10}$

New example:

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

Calculate $\mathbf{A}^2$

$$\mathbf{A}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}$$

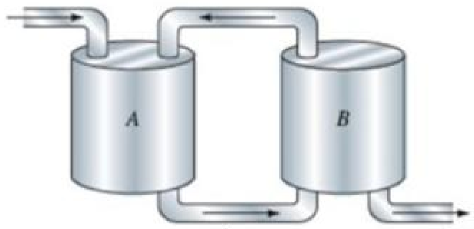
Calculate $\mathbf{A}^3$

$$\mathbf{A}^3 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix} = \begin{pmatrix} 2^3 & 0 \\ 0 & 3^3 \end{pmatrix}$$

Calculate $\mathbf{A}^{10}$

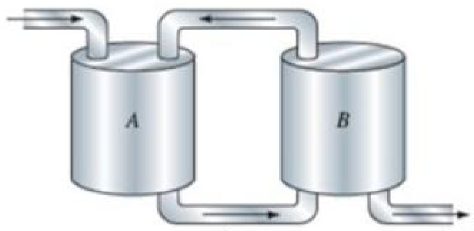
$$\mathbf{A}^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{pmatrix}$$

Let $x(t)$ be the amount of salt in the first tank
Let $y(t)$ be the amount of salt in the second tank

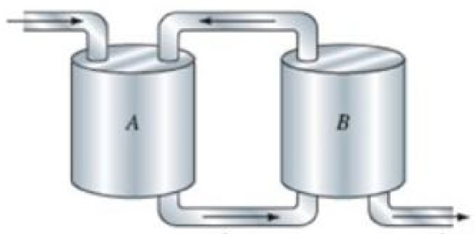


$$\frac{dx}{dt} = -\frac{1}{2}x + \frac{1}{8}y$$

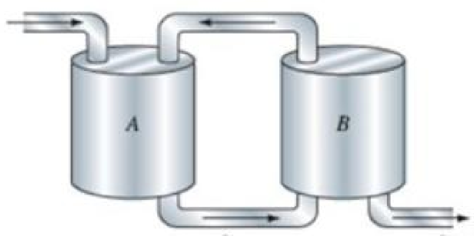
$$\frac{dy}{dt} = \quad \frac{1}{2}x - \frac{1}{2}y$$



$$\begin{pmatrix} x' \\ \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x + \frac{1}{8}y \\ \\ \frac{1}{2}x - \frac{1}{2}y \end{pmatrix}$$

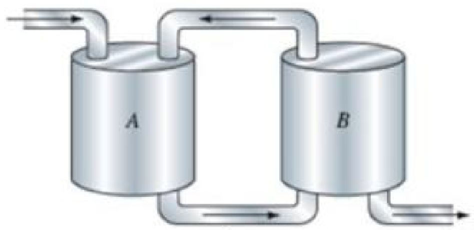


$$\begin{pmatrix} x' \\ \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x + \frac{1}{8}y \\ \\ \frac{1}{2}x - \frac{1}{2}y \end{pmatrix} = \begin{pmatrix} -1/2 & 1/8 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\text{Let} \quad \vec{\mathbf{X}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$$



$$\text{Let} \quad \vec{\mathbf{X}} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Suppose $\mathbf{A}$ is a diagonal matrix.

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Solve the equation:

$$\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

$$x' = ax \qquad\qquad y' = by$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$$

$$x' = ax \qquad\qquad y' = by$$

$$x(t) = x_0 e^{at} \qquad\qquad y(t) = y_0 e^{bt}$$

Review from last time:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix} \quad \text{had eigenvectors:} \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \qquad \mathbf{P}^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$
\begin{aligned}
\mathbf{P}^{-1}\mathbf{AP} &= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 9 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}
\end{aligned}
$$

Can we always do this?

How can we exploit this to solve problems?

Suppose **A** is an n by n matrix and we have calculated eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, ... $\vec{\mathbf{v}}_n$. Form a matrix **P** where each eigenvector is a column of **P**.

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

Will this always result in a diagonal matrix?

Suppose $\mathbf{A}$ is an n by n matrix and we have calculated eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\ldots$ $\vec{\mathbf{v}}_n$. Form a matrix $\mathbf{P}$ where each eigenvector is a column of $\mathbf{P}$.

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

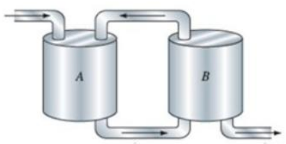
Will this always result in a diagonal matrix?

**Warning:**
This formula assumes that $\mathbf{P}$ has an inverse.

How can we exploit this?

$$\begin{pmatrix} x' \\ \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x + \frac{1}{8}y \\ \\ \frac{1}{2}x - \frac{1}{2}y \end{pmatrix} = \begin{pmatrix} -1/2 & 1/8 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{Let} \quad \vec{\mathbf{X}} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$$



Begin by finding the eigenvalues and eigenvectors:

$$\mathbf{A} = \begin{pmatrix} -1/2 & 1/8 \\ 1/2 & -1/2 \end{pmatrix}$$

$$\text{Solve:} \qquad \det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

$$\lambda_1 = -\frac{1}{4} \qquad \lambda_2 = -\frac{3}{4}$$

$$\text{Eigenvectors:} \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} -1/2 & 1/8 \\ 1/2 & -1/2 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix}$$

Let $\mathbf{V} = \mathbf{P^{-1}}\vec{\mathbf{X}}$
We can now substitute $\vec{\mathbf{X}} = \mathbf{P}\vec{\mathbf{V}}$ into $\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$

$$\frac{d}{dt}(\mathbf{P}\vec{\mathbf{V}}) = \mathbf{A}\mathbf{P}\vec{\mathbf{V}}$$

$$\frac{d}{dt}(\mathbf{P}\vec{\mathbf{V}}) = \mathbf{AP}\vec{\mathbf{V}}$$

$$\mathbf{P}\frac{d\vec{\mathbf{V}}}{dt} = \mathbf{AP}\vec{\mathbf{V}}$$

$$\frac{d\vec{\mathbf{V}}}{dt} = \mathbf{P}^{-1}\mathbf{AP}\vec{\mathbf{V}}$$

$$\frac{d\vec{\mathbf{V}}}{dt} = \begin{pmatrix} -1/4 & 0 \\ 0 & -3/4 \end{pmatrix}\vec{\mathbf{V}}$$

$$\frac{d\vec{\mathbf{V}}}{dt} = \begin{pmatrix} -1/4 & 0 \\ 0 & -3/4 \end{pmatrix} \vec{\mathbf{V}}$$

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} -1/4 & 0 \\ 0 & -3/4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} -v_1/4 \\ -3v_2/4 \end{pmatrix}$$

$$\frac{dv_1}{dt} = -\frac{1}{4}v_1 \qquad \frac{dv_2}{dt} = -\frac{3}{4}v_2$$

$$\frac{dv_1}{dt} = -\frac{1}{4}v_1 \qquad \frac{dv_2}{dt} = -\frac{3}{4}v_2$$

$$v_1 = ae^{-t/4} \qquad v_2 = be^{-3t/4}$$

$$v_1 = ae^{-t/4} \qquad v_2 = be^{-3t/4}$$

$$\vec{\mathbf{X}} = \mathbf{P}\vec{\mathbf{V}}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} ae^{-t/4} \\ be^{-3t/4} \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ae^{-t/4} + be^{-3t/4} \\ 2ae^{-t/4} - 2be^{-3t/4} \end{pmatrix}$$

$$x(t) = ae^{-t/4} + be^{-3t/4}$$

$$y(t) = 2ae^{-t/4} - 2be^{-3t/4}$$