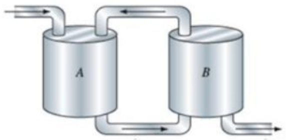
Differential Equations and Matrix Methods
Dr. E. Jacobs

Today's Topic : Matrix Differential Equations

$$\begin{pmatrix} x' \\ \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x + \frac{1}{8}y \\ \\ \frac{1}{2}x - \frac{1}{2}y \end{pmatrix} = \begin{pmatrix} -1/2 & 1/8 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{Let} \quad \vec{\mathbf{X}} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$$



$$\mathbf{A} = \begin{pmatrix} -1/2 & 1/8 \\ 1/2 & -1/2 \end{pmatrix}$$

$$\lambda_1 = -\frac{1}{4} \qquad \lambda_2 = -\frac{3}{4}$$

Eigenvectors: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\mathbf{A} = \begin{pmatrix} -1/2 & 1/8 \\ 1/2 & -1/2 \end{pmatrix}$$

$$\lambda_1 = -\frac{1}{4} \qquad \lambda_2 = -\frac{3}{4}$$

Eigenvectors: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ae^{-t/4} + be^{-3t/4} \\ 2ae^{-t/4} - 2be^{-3t/4} \end{pmatrix}$$

$$
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ae^{-t/4} + be^{-3t/4} \\ 2ae^{-t/4} - 2be^{-3t/4} \end{pmatrix}
$$

$$
= \begin{pmatrix} ae^{-t/4} \\ 2ae^{-t/4} \end{pmatrix} + \begin{pmatrix} be^{-3t/4} \\ -2be^{-3t/4} \end{pmatrix}
$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ae^{-t/4} + be^{-3t/4} \\ 2ae^{-t/4} - 2be^{-3t/4} \end{pmatrix}$$
$$= \begin{pmatrix} ae^{-t/4} \\ 2ae^{-t/4} \end{pmatrix} + \begin{pmatrix} be^{-3t/4} \\ -2be^{-3t/4} \end{pmatrix}$$
$$= a \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/4} + b \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t/4}$$

The general solution is a linear combination of vectors of the form $\vec{\mathbf{u}}e^{rt}$

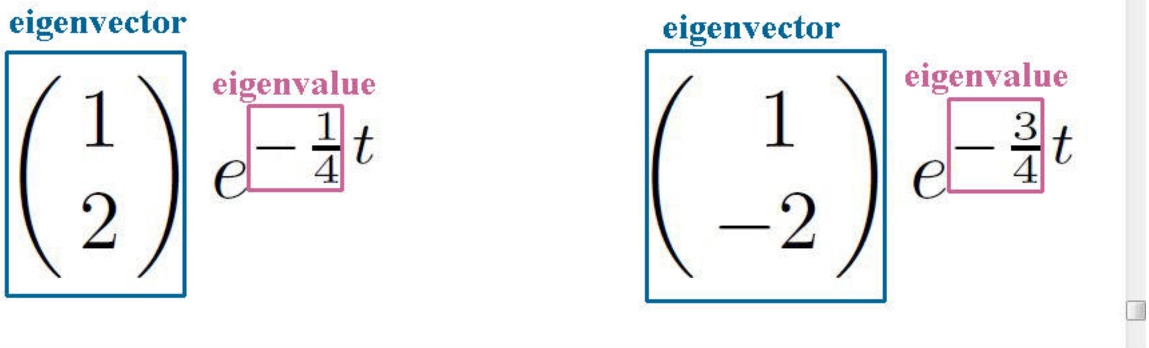
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/4} + b \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t/4}$$

The general solution is a linear combination of vectors of the form $\vec{\mathbf{u}} e^{rt}$

How can you predict $\vec{\mathbf{u}}$ and r?

The general solution is a linear combination of vectors of the form $\vec{\mathbf{u}}e^{rt}$

Each r is an eigenvalue and $\vec{\mathbf{u}}$ is an eigenvector



Example: Find two functions $x_1 = x_1(t)$ and $x_2(t)$ such that:

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= -x_1 + 4x_2 \end{aligned}$$

Initial conditions: $x_1(0) = 1$ and $x_2(0) = 0$

Find a vector $\vec{\mathbf{X}} = \binom{x_1}{x_2}$ such that:

$$\frac{d\vec{\mathbf{X}}}{dt} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \vec{\mathbf{X}}$$

Initial condition: $\vec{\mathbf{X}} = \binom{1}{0}$

We expect that the general solution will be a linear combination of vectors of the form $\vec{\mathbf{u}}e^{rt}$

$$\frac{d\vec{\mathbf{X}}}{dt} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \vec{\mathbf{X}}$$

Eigenvalues:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda_1 = 2 \text{ and } \lambda_2 = 3$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \vec{\mathbf{X}}$$

Eigenvalues	$\lambda_1 = 2$	$\lambda_2 = 3$
Eigenvectors	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\frac{d\vec{\mathbf{X}}}{dt} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \vec{\mathbf{X}}$$

Eigenvalues $\lambda_1 = 2$ $\lambda_2 = 3$

Eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{\mathbf{X}}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$\vec{\mathbf{X}}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

If $\vec{\mathbf{X}}(0) = \binom{1}{0}$ then:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ c_1 + c_2 \end{pmatrix}$$

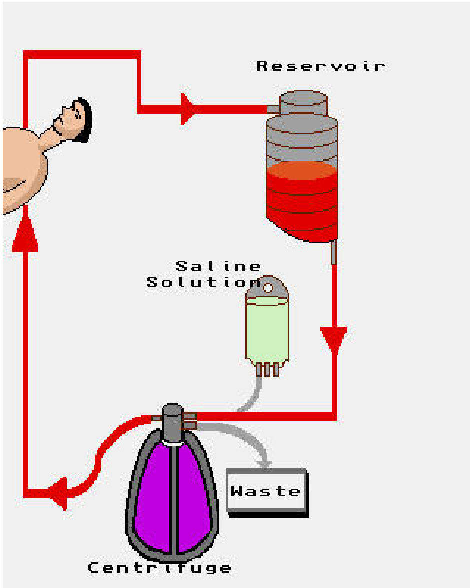
$$1 = 2c_1 + c_2 \qquad 0 = c_1 + c_2$$

$$c_1 = 1 \qquad c_2 = -1$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + (-1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$x_1(t) = 2e^{2t} - e^{3t} \qquad x_2(t) = e^{2t} - e^{3t}$$

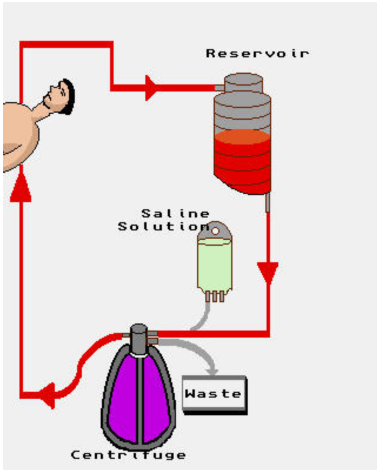
The Intravenous Drug Problem



Let $x(t)$ denote the number of milligrams of medicine in the patient after t hours. Assume $x(0) = 0$

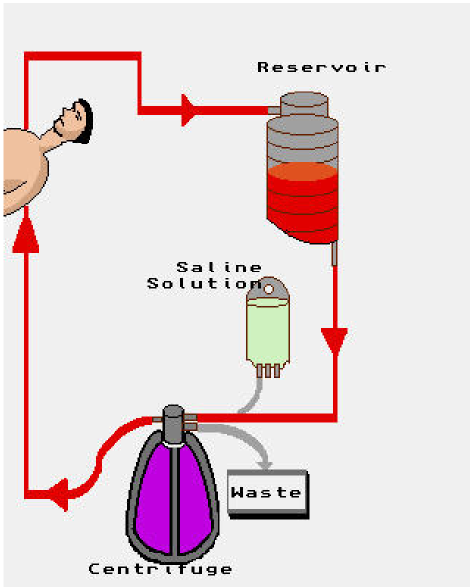
Let $y(t)$ denote the number of milligrams of medicine in the external system after t hours.

Assume $y(0) = 1$ mg.



Assumptions:

1. Patient's bloodstream holds 5,000 ml of blood.
2. External system holds 1,025 ml of liquid.
3. Blood is drawn from the patient at 200 ml/hour.
4. Blood is infused into the patient at 200 ml/hour.
5. Waste is removed from the system at 6 ml/hour.
6. Saline is added to the system at 6 ml/hour



$$\begin{array}{c}\text{Rate out}\\ \text{of patient}\end{array} = \left(200\,\frac{\text{ml}}{\text{hour}}\right)\cdot\left(\frac{x}{5000}\,\frac{\text{mg}}{\text{ml}}\right)$$

$$\begin{array}{c}\text{Rate into}\\ \text{patient}\end{array} = \left(200\,\frac{\text{ml}}{\text{hour}}\right)\left(\frac{y}{1025}\,\frac{\text{mg}}{\text{ml}}\right)$$

$$\frac{dx}{dt} = -(200)\cdot\frac{x}{5000} + (200)\cdot\frac{y}{1025}$$

$$\frac{dx}{dt} = -\frac{1}{25}x + \frac{8}{41}y$$

$$\begin{array}{c}\text{Rate into}\\\text{Ext. sys.}\end{array} = \left(200\,\frac{\text{ml}}{\text{hour}}\right)\cdot\left(\frac{x}{5000}\,\frac{\text{mg}}{\text{ml}}\right)$$

$$\begin{array}{c}\text{Rate out of}\\\text{Ext. Sys.}\end{array} = \left(200\,\frac{\text{ml}}{\text{hour}}\right)\cdot\left(\frac{y}{1025}\,\frac{\text{mg}}{\text{ml}}\right) + \left(6\frac{\text{ml}}{\text{hour}}\right)\left(\frac{y}{1025}\,\frac{\text{mg}}{\text{ml}}\right)$$

$$\frac{dy}{dt} = \frac{200x}{5000} - \frac{200y}{1025} - \frac{6y}{1025}$$

$$\frac{dy}{dt} = \frac{1}{25}x - \frac{206}{1025}y$$

$$\frac{dx}{dt} = -\frac{1}{25}x + \frac{8}{41}y$$
$$\frac{dy}{dt} = \frac{1}{25}x - \frac{206}{1025}y$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{1}{25} & \frac{8}{41} \\ \\ \frac{1}{25} & -\frac{206}{1025} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$$

$$\text{where} \quad \vec{\mathbf{X}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\text{and} \quad \mathbf{A} = \begin{pmatrix} -\frac{1}{25} & \frac{8}{41} \\ \\ \frac{1}{25} & -\frac{206}{1025} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{25} & \frac{8}{41} \\ \\ \frac{1}{25} & -\frac{206}{1025} \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + \frac{247}{1025}\lambda + \frac{6}{25,625} = 0$$

$$\lambda_1 = -\frac{6}{25} \qquad \lambda_2 = -\frac{1}{1,025}$$

Eigenvectors:

$$\text{For} \quad \lambda_1 = -\frac{6}{25} \qquad \vec{\mathbf{v}}_1 = \begin{pmatrix} -40 \\ 41 \end{pmatrix}$$

$$\text{For} \quad \lambda_2 = -\frac{1}{1,025} \qquad \vec{\mathbf{v}}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\vec{\mathbf{X}} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -40 \\ 41 \end{pmatrix} e^{-\frac{6t}{25}} + c_2 \begin{pmatrix} 5 \\ 1 \end{pmatrix} e^{-\frac{t}{1,025}}$$

$$x(t) = -40c_1e^{-\frac{6t}{25}} + 5c_2e^{-\frac{t}{1{,}025}}$$

$$y(t) = 41c_1e^{-\frac{6t}{25}} + c_2e^{-\frac{t}{1{,}025}}$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$$