Differential Equations and Matrix Methods
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Today's Topic : Higher Order Differential Equations

$$\vec{\mathbf{X}} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Solve the equation $\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$

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Solve the equation $\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$

We anticipate that the general solution will be a linear combinations of vectors of the form $\vec{\mathbf{u}}e^{rt}$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Find the eigenvalues by solving $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + \lambda = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + \lambda = 0$$

$$-\lambda(\lambda - 1)(\lambda + 1) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = -1$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda = 0 \qquad \lambda = 1 \qquad \lambda = -1$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Solve the equation $\frac{d\vec{\mathbf{X}}}{dt} = \mathbf{A}\vec{\mathbf{X}}$

Solution:

$$\vec{\mathbf{X}} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{1t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-1t}$$

First order equations:

$$\frac{dy}{dt} + P(t)y = Q(t)$$

Systems of first order equations:

$$x' = a_{11}x + a_{12}y$$
$$y' = a_{21}x + a_{22}y$$

Higher order equations:

$$\frac{d^2y}{dt^2} + y = 0$$

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$$

$$\frac{d^3y}{dt^3} - \frac{dy}{dt} = 0$$

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$$y''' - y' = 0$$

$$y''' = y'$$

Let $x_1 = y \qquad x_2 = y' \qquad x_3 = y''$

$$\vec{\mathbf{X}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

$$y''' = y'$$

Let $x_1 = y$ $\qquad x_2 = y'$ $\qquad x_3 = y''$

$$\vec{\mathbf{X}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_2 \end{pmatrix}$$

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Let $x_1 = y \qquad x_2 = y' \qquad x_3 = y''$

$$\vec{\mathbf{X}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

$$\vec{\mathbf{X}} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{1t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-1t}$$

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$$\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} c_1 + c_2 e^{t} + c_3 e^{-t} \\ c_2 e^{t} - c_3 e^{-t} \\ c_2 e^{t} + c_3 e^{-t} \end{pmatrix}$$

$$\vec{\mathbf{X}} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{1t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-1t}$$

$$\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} c_1 + c_2 e^t + c_3 e^{-t} \\ c_2 e^t - c_3 e^{-t} \\ c_2 e^t + c_3 e^{-t} \end{pmatrix}$$

$$y = c_1 + c_2 e^t + c_3 e^{-t}$$

The general solution of $y''' - y' = 0$ is:

$$y = c_1 + c_2 e^t + c_3 e^{-t}$$

Notice that the general solution is a linear combination of expressions of the form e^{rt}

$$y''' - y' = 0$$

Short-cut : Look for all solutions of the form e^{rt}

$$y = e^{rt} \qquad y' = re^{rt} \qquad y'' = r^2 e^{rt} \qquad y''' = r^3 e^{rt}$$

Substitute into the equation $y''' - y' = 0$

$$r^3 e^{rt} - re^{rt} = 0$$

$$r^3 - r = 0$$

$$y''' - y' = 0$$

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$y = e^{rt}$ $\qquad y' = re^{rt}$ $\qquad y'' = r^2e^{rt}$ $\qquad y''' = r^3e^{rt}$

Substitute into the equation $y''' - y' = 0$

$$r^3e^{rt} - re^{rt} = 0$$

$$r(r-1)(r+1) = r^3 - r = 0$$

$$y''' - y' = 0$$

Short-cut: Look for all solutions of the form e^{rt}

$$r(r-1)(r+1) = r^3 - r = 0$$

Solutions: $r = 0$ $\qquad r = 1$ $\qquad r = -1$

$$y = c_1 e^{0t} + c_2 e^{1t} + c_3 e^{-1t}$$

Solve the equation: $$\frac{d^2y}{dt^2} - 4y = 0$$

We anticipate that the answer will be a linear combination of terms of the form e^{rt} so substitute $y = e^{rt}$ and solve for r

$$r^2 e^{rt} - 4e^{rt} = 0$$

$$r^2 - 4 = 0 \qquad \text{so} \qquad r = \pm 2$$

$$y = c_1 e^{2t} + c_2 e^{-2t}$$

$$\frac{d^2y}{dt^2} - 4y = 0$$

$$y'' = 4y$$

$$\text{Let } \vec{\mathbf{X}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\frac{d^2y}{dt^2} - 4y = 0$$

$$y'' = 4y$$

$$\text{Let } \vec{\mathbf{X}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ 4y \end{pmatrix} = \begin{pmatrix} x_2 \\ 4x_1 \end{pmatrix}$$

$$\frac{d^2y}{dt^2} - 4y = 0$$

$$y'' = 4y$$

$$\text{Let } \vec{\mathbf{X}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\frac{d\vec{\mathbf{X}}}{dt} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} x_2 \\ 4x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find $\vec{\mathbf{X}} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ that solves the equation:

$$\frac{d\vec{\mathbf{X}}}{dt} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \vec{\mathbf{X}}$$

Find the eigenvalues and eigenvectors of $\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$

$$\det \begin{pmatrix} 0 - r & 1 \\ 4 & 0 - r \end{pmatrix} = r^2 - 4 = 0$$

$$r = \pm 2$$

$$\vec{\mathbf{X}} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$$

$$\vec{\mathbf{X}} = \begin{pmatrix} c_1 e^{2t} + c_2 e^{-2t} \\ 2c_1 e^{2t} - 2c_2 e^{-2t} \end{pmatrix}$$

Since $\vec{\mathbf{X}} = \begin{pmatrix} y \\ y' \end{pmatrix}$, the solution of $y'' - 4y = 0$ is:

$$y = c_1 e^{2t} + c_2 e^{-2t}$$

Theorem: If y_1, y_2, y_3, ..., y_n are n linearly independent solutions of the differential equation:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_1 \frac{dy}{dx} + a_0 y = 0$$

then the general solution is:

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n$$