Differential Equations
Dr. E. Jacobs

The Exponential Shift Theorem

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

$$(D^2 + 3D + 2)y = 0$$

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Look for solutions of the form e^{rx}

$$\left(D^2 + 3D + 2\right)\left(e^{rx}\right) = 0$$

$$r^2 e^{rx} + 3re^{rx} + 2e^{rx} = 0$$

$$r^2 + 3r + 2 = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$$r = -1 \qquad \text{and} \qquad r = -2$$

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

$$(D^2 + 2D + 1)y = 0$$

Look for solutions of the form e^{rx}

$$r^2 + 2r + 1 = 0$$

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$$(r + 1)(r + 1) = 0$$

$$r = -1$$

$$(D^2 + 2D + 1)y = 0$$

The only solution of the form $y = e^{rx}$ is e^{-x}. The general solution is:

$$y = c_1 e^{-x} + c_2 \left(\quad ? \quad \right)$$

$$\frac{d}{dx}(g(x)f(x)) = g(x)f'(x) + g'(x)f(x)$$

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$$D\left(e^{rx}f(x)\right) = e^{rx}f'(x) + re^{rx}f(x)$$

$$D\left(e^{rx}f(x)\right) = e^{rx}(D + r)f(x)$$

$$D\left(e^{rx}f(x)\right) = e^{rx}(D+r)f(x)$$

$$\begin{aligned} D^2\left(e^{rx}f(x)\right) &= D(D\left(e^{rx}f(x)\right) \\ &= D\left(e^{rx}(D+r)f(x)\right) \end{aligned}$$

Replace $f(x)$ with $(D+r)f(x)$

$$D\left(e^{rx}f(x)\right) = e^{rx}(D+r)f(x)$$

$$D\left(e^{rx}(D+r)f(x)\right) = e^{rx}(D+r)(D+r)f(x)$$

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$$\begin{aligned}
D^2\left(e^{rx}f(x)\right) &= D(D\left(e^{rx}f(x)\right)) \\
&= D\left(e^{rx}(D+r)f(x)\right) \\
&= e^{rx}(D+r)(D+r)f(x) \\
&= e^{rx}(D+r)^2 f(x)
\end{aligned}$$

$$
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D\left(e^{rx}f(x)\right) &= e^{rx}(D+r)f(x)\\
D^2\left(e^{rx}f(x)\right) &= e^{rx}(D+r)^2f(x)\\
D^3\left(e^{rx}f(x)\right) &= e^{rx}(D+r)^3f(x)\\
&\vdots\\
D^k\left(e^{rx}f(x)\right) &= e^{rx}(D+r)^kf(x)
\end{aligned}
$$

Example.
If $y = x^4 e^x$, calculate the third derivative.

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Solution:

$$\begin{aligned} D^3 y &= D^3 \left(x^4 e^x \right) \\ &= e^x (D+1)^3 x^4 \\ &= e^x (D^3 + 3D^2 + 3D + 1)(x^4) \\ &= e^x (24x + 36x^2 + 12x^3 + x^4) \end{aligned}$$

Let $P(t)$ be the following polynomial:

$$\begin{aligned} P(t) &= a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \\ &= \sum_{k=0}^{n} a_k t^k \end{aligned}$$

$P(D)$ stands for the derivative operator:

$$a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

In summation notation:

$$P(D) = \sum_{k=0}^{n} a_k D^k$$

$$
\begin{aligned}
P(D)\left(e^{rx}f(x)\right) &= \sum_{k=0}^{n} a_k D^k \left(e^{rx}f(x)\right) \\
&= \sum_{k=0}^{n} a_k e^{rx}(D+r)^k f(x) \\
&= e^{rx}\sum_{k=0}^{n} a_k (D+r)^k f(x) \\
&= e^{rx}P(D+r)f(x)
\end{aligned}
$$

The Exponential Shift Theorem

$$P(D)\left(e^{rx}f(x)\right) = e^{rx}P(D+r)f(x)$$

Example
Let $y = e^{-x}\sin x$. Calculate the expression $y'' + y'$

Solution:

$$\begin{aligned}(D^2 + D)y &= (D^2 + D)\left(e^{-x}\sin x\right)\\ &= e^{-x}\left((D-1)^2 + (D-1)\right)(\sin x)\\ &= e^{-x}\left(D^2 - D\right)(\sin x)\\ &= e^{-x}(-\sin x - \cos x)\end{aligned}$$

Example

Solve the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$y = c_1 e^{-x} + c_2 \left(\quad ? \quad \right)$$

Let $u = e^x y$. Therefore, $y = e^{-x} u$

$$(D+1)^2 y = 0$$

$$(D+1)^2 \left(e^{-x} u\right) = 0$$

$$e^{-x} D^2 u = 0$$

$$D^2 u = 0$$

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$$D^2 u = 0$$

$$Du = C_1$$

Let $u = e^x y$. Therefore, $y = e^{-x} u$

$$D^2 u = 0$$

$$Du = C_1$$

$$u = C_1 x + C_2$$

$$y = e^{-x} u = C_1 x e^{-x} + C_2 e^{-x}$$

Example

Solve the differential equation:

$$(D-4)^3 y = 0$$

$$y = ae^{4x} + b(\quad ? \quad) + c(\quad ? \quad)$$

Let $u = e^{-4x}y$ and substitute into the equation.

$$(D-4)^3\left(e^{4x}u\right) = 0$$

$$e^{4x}D^3u = 0$$

$$D^3u = 0$$

Now, integrate both sides three times to obtain:

$$u = a + bx + cx^2$$

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$$y = e^{4x}u = ae^{4x} + bxe^{4x} + cx^2e^{4x}$$