Differential Equations
Dr. E. Jacobs

Today's Topic : The Repeated Root Case

Review:

$$(D-4)^3 y = 0$$

Solution:

$$y = c_1 e^{4x} + c_2 x e^{4x} + c_3 x^2 e^{4x}$$

The solution is a linear combination of:

$$e^{4x} \qquad x e^{4x} \qquad x^2 e^{4x}$$

$$(D-4)^5 y = 0$$

Solution:

$$y = c_1 e^{4x} + c_2 x e^{4x} + c_3 x^2 e^{4x} + c_4 x^3 e^{4x} + c_5 x^4 e^{4x}$$

The solution is a linear combination of:

$e^{4x} \qquad x e^{4x} \qquad x^2 e^{4x} \qquad x^3 e^{4x} \qquad x^4 e^{4x}$

$$(D-4)^5 y = 0$$

Solution:

$$y = c_1 e^{4x} + c_2 x e^{4x} + c_3 x^2 e^{4x} + c_4 x^3 e^{4x} + c_5 x^4 e^{4x}$$

The solution is a linear combination of:

e^{4x} $\quad xe^{4x}$ $\quad x^2 e^{4x}$ $\quad x^3 e^{4x}$ $\quad x^4 e^{4x}$

$$(D-r)^n y = 0$$

Solution:

$$y = c_1 e^{rx} + c_2 x e^{rx} + c_3 x^2 e^{rx} + \cdots + c_n x^{n-1} e^{rx}$$

The solution is a linear combination of:

$$e^{rx} \qquad x e^{rx} \qquad x^2 e^{rx} \qquad \cdots \qquad x^{n-1} e^{rx}$$

Solve:

$$\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0$$

$$(D^4 - 2D^2 + 1)y = 0$$

$$(D^2 - 1)(D^2 - 1)y = 0$$

$$(D - 1)(D + 1)(D - 1)(D + 1)y = 0$$

$$(D - 1)^2(D + 1)^2y = 0$$

Start with:

$$(D+1)^2 y = 0$$

Any linear combination of e^{-x} and xe^{-x} will be a solution of $(D+1)^2 y = 0$. Therefore, any linear combination of these two functions will also be a solution of:

$$(D-1)^2(D+1)^2 y = 0$$

$$(D+1)^2 y = 0$$

$$(D-1)^2 (D+1)^2 y = (D-1)^2 (0)$$

$$(D-1)^2 (D+1)^2 y = (D^2 - 2D + 1)(0)$$

$$(D-1)^2 (D+1)^2 y = 0$$

$$(D-1)^2(D+1)^2y = 0$$

$$(D+1)^2(D-1)^2y = 0$$

This time, start with $(D-1)^2y = 0$.

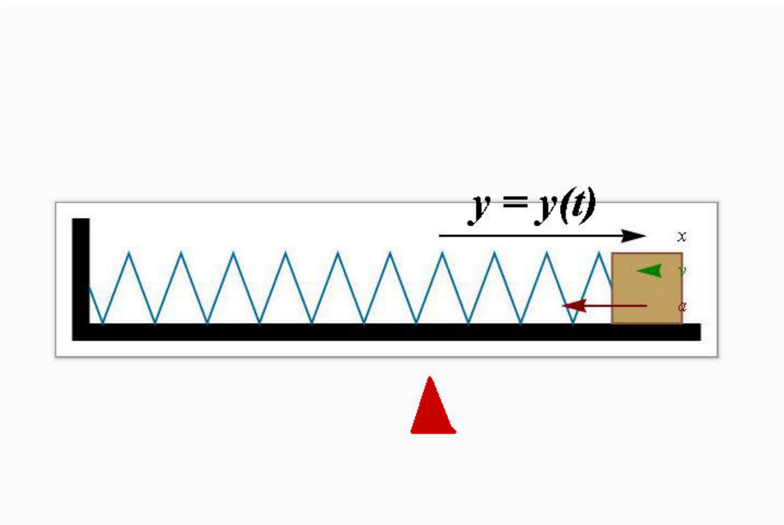
Any linear combination of e^x and xe^x will be a solution of $(D-1)^2y = 0$. Therefore, any linear combination of these two functions will also be a solution of:

$$(D+1)^2(D-1)^2y = 0$$

The solution of $(D+1)^2(D-1)^2y=0$ is

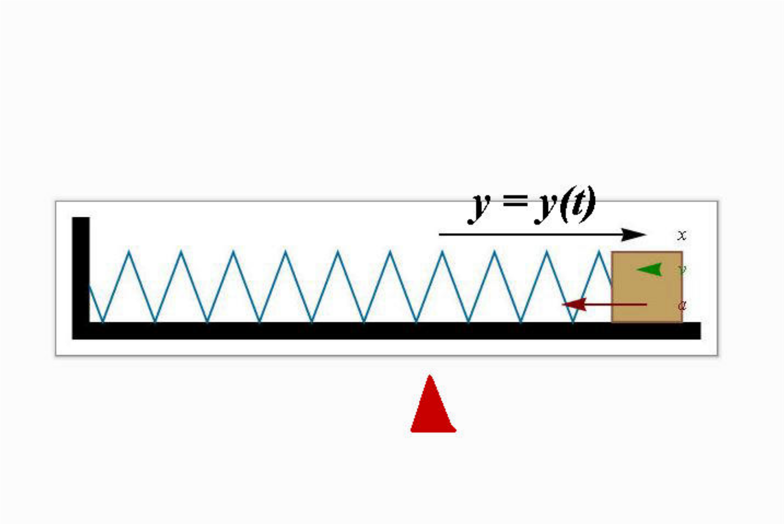
$$y = c_1e^{x} + c_2xe^{x} + c_3e^{-x} + c_4xe^{-x}$$

Spring Motion:



Hooke's Law: The force of the spring is proportional to the displacement of the object.

$$F_s = -ky$$



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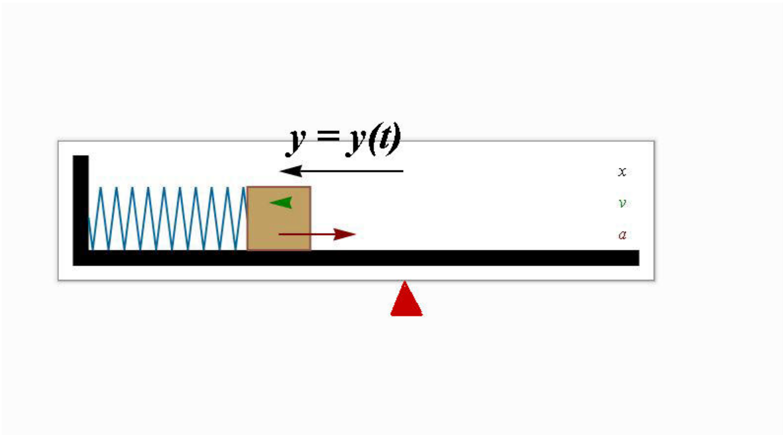
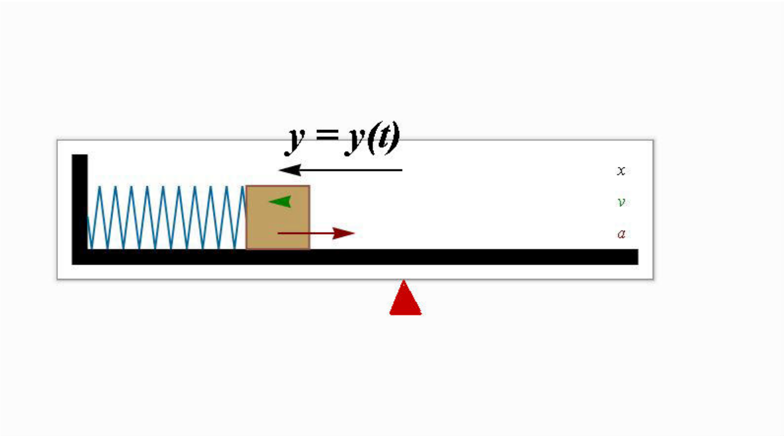


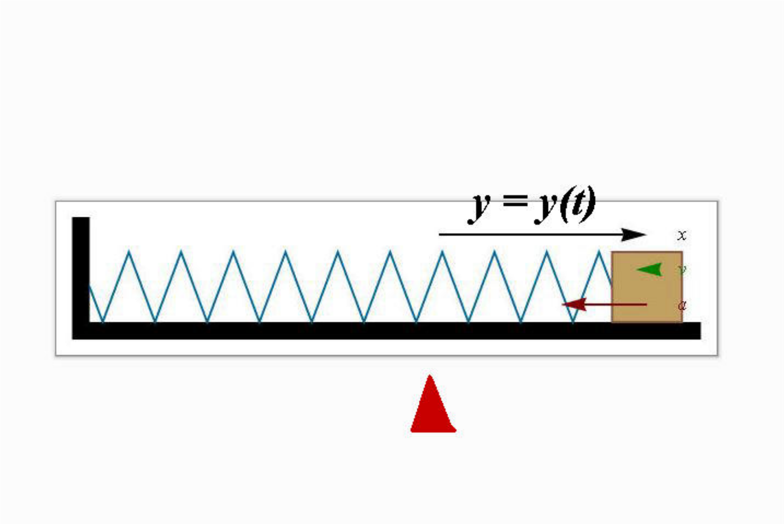
Resistive (damping) force is proportional to the velocity of the object

$$F_r = -\beta v$$



$$(\text{mass})(\text{acceleration}) = \text{Net force}$$

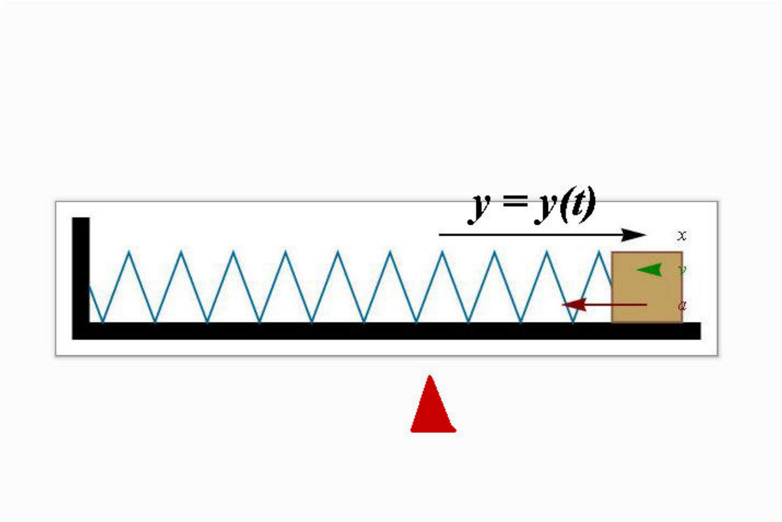
$$ma = F_s + F_r$$



$$ma = F_s + F_r$$

$$m\frac{d^2y}{dt^2} = -ky - \beta\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Suppose $m = 1$, $\beta = 4$ and $k = 4$.
Initial conditions: $y(0) = 1$, $y'(0) = 0$

$$(D^2 + 4D + 4)y = 0$$

$$(D^2 + 4D + 4)y = 0$$

$$(D + 2)^2 y = 0$$

$$y = c_1 e^{-2t} + c_2 t e^{-2t}$$

$y(0) = 1$ implies that $c_1 = 1$

$$y = e^{-2t} + c_2 t e^{-2t}$$

$$y' = -2e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$y' = 0$ when $t = 0$

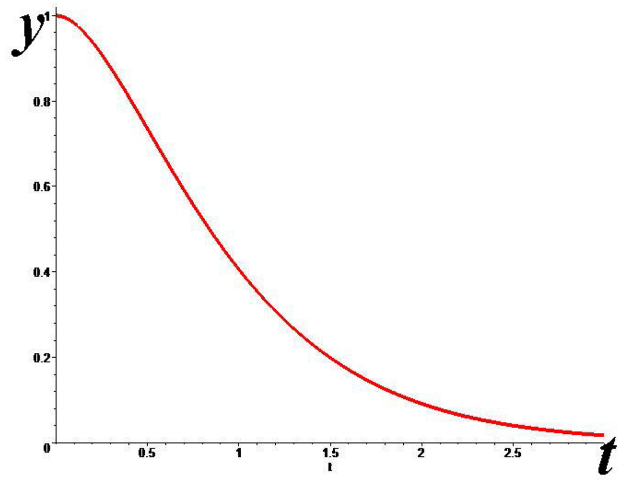
$$0 = -2 + c_2$$

$$c_2 = 2$$

$$y = e^{-2t} + 2te^{-2t}$$

$$(D+2)^2 y = 0$$

$$y = e^{-2t} + 2te^{-2t}$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Suppose $m = 1$, $\beta = 0$ and $k = 4$.

$$\frac{d^2y}{dt^2} + 4y = 0$$

$$\frac{d^2y}{dt^2} + 4y = 0$$

$$(D^2 + 4)y = 0$$

Substitute $y = e^{rt}$

$$r^2 + 4 = 0$$

$$r^2 = -4$$

What does e^{rt} mean if r is an imaginary number?