Differential Equations

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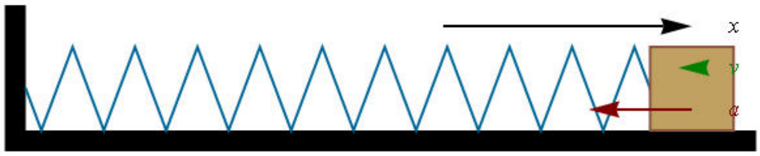
Today's Topic : The Spring Equation

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Substitute $y = e^{rt}$

$$mr^2 + \beta r + k = 0$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Substitute $y = e^{rt}$

$$mr^2 + \beta r + k = 0$$

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$$

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Case 1. (Overdamped Case)

Suppose $\beta^2 - 4mk > 0$

$$r_1 = \frac{-\beta - \sqrt{\beta^2 - 4mk}}{2m} \qquad r_2 = \frac{-\beta + \sqrt{\beta^2 - 4mk}}{2m}$$

Both r_1 and r_2 are negative numbers

$$\beta^2 - 4mk < \beta^2$$

$$\sqrt{\beta^2 - 4mk} < \beta$$

$$-\beta + \sqrt{\beta^2 - 4mk} < 0$$

$$r_2 = \frac{-\beta + \sqrt{\beta^2 - 4mk}}{2m} < 0$$

Case 1. (Overdamped Case)

Suppose $\beta^2 - 4mk > 0$

$$r_1 = \frac{-\beta - \sqrt{\beta^2 - 4mk}}{2m} \qquad r_2 = \frac{-\beta + \sqrt{\beta^2 - 4mk}}{2m}$$

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$\lim_{t \to \infty} y(t) = 0$$

Case 2. (Critically Damped Case)

Suppose $\beta^2 - 4mk = 0$

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m} = -\frac{\beta}{2m}$$

This is the repeated root case

$$y(t) = c_1 e^{-\frac{\beta}{2m}t} + c_2 t e^{-\frac{\beta}{2m}t}$$

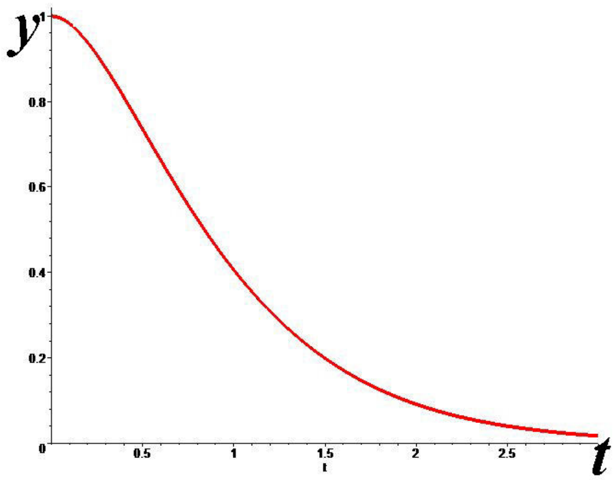
$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Suppose $m = 1$, $\beta = 4$ and $k = 4$.
Initial conditions: $y(0) = 1$, $y'(0) = 0$

$$\beta^2 - 4mk = 4^2 - 4(1)(4) = 0$$

$$y = e^{-2t} + 2te^{-2t}$$

$$y(t) = e^{-2t} + 2te^{-2t}$$



Case 3. (Underdamped Case)

Suppose $\beta^2 - 4mk < 0$

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m} \quad \longleftarrow \text{Complex number}$$

Case 3. (Underdamped Case)

Suppose $\beta^2 - 4mk < 0$

$$
\begin{aligned}
r &= \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m} \\
&= \frac{-\beta \pm \sqrt{(4mk - \beta^2)(-1)}}{2m} \\
&= \frac{-\beta \pm i\sqrt{4mk - \beta^2}}{2m}
\end{aligned}
$$

$$r = \frac{-\beta \pm i\sqrt{4mk - \beta^2}}{2m}$$

$$\text{Let} \quad \omega = \frac{\sqrt{4mk - \beta^2}}{2m}$$

$$r = -\frac{\beta}{2m} \pm \omega i$$

$$y(t) = c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t$$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$

Suppose $m = 1$, $\beta = 4$ and $k = 20$

$$\beta^2 - 4mk = 16 - (4)(1)(20) = -64 < 0$$

With the initial conditions $y(0) = 0$ and $y'(0) = 8$, we obtained:

$$y(t) = 2e^{-2t}\sin 4t$$

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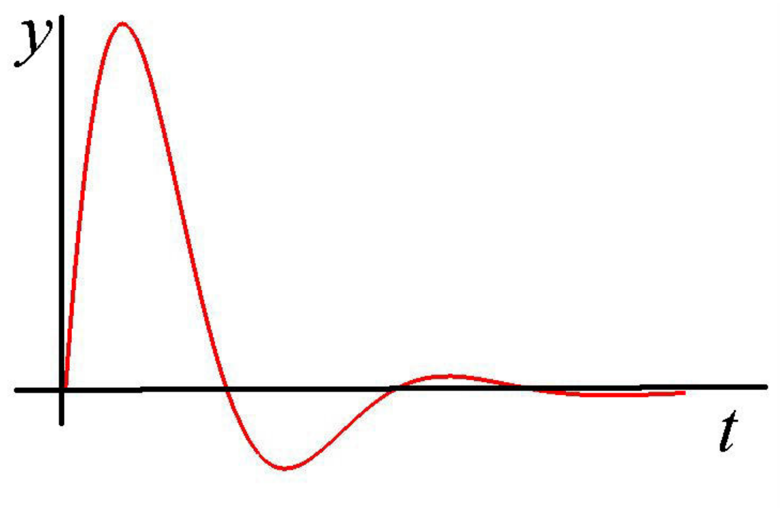

Underdamped Case:

$$y(t) = c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t$$