# Differential Equations
Dr. E. Jacobs

## Today's Topic : Nonhomogeneous Equations

Homogeneous Equation:

$$P(D)y = 0$$

Nonhomogeneous Equation:

$$P(D)y = f$$

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

$$P(D) = \sum_{k=0}^{\infty} a_k D^k$$

Homogeneous Linear Equation: $P(D)y = 0$

$$(D + 2)\, y = 0$$

$$\left(D^2 + 3D + 2\right) y = 0$$

$$\left(D^3 - 3D^2 + 3D - 1\right) y = 0$$

Homogeneous Linear Equation: $P(D)y = 0$

$$\frac{dy}{dx} + 2y = 0$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$$

Nonhomogeneous Equation: $P(D)y = f$

$$\frac{dy}{dx} + 2y = 2x$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin x$$

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = -e^x$$

Nonhomogeneous Equation: $P(D)y = f$

$$\frac{dy}{dx} + 2y - 2x = 0$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y - \sin x = 0$$

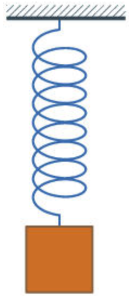
$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y + e^x = 0$$

Spring Motion:

$$(mD^2 + \beta D + k)y(t) = 0$$

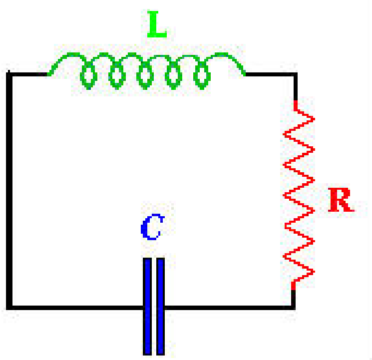
Add in the force of gravity

$$(mD^2 + \beta D + k)y(t) = mg$$



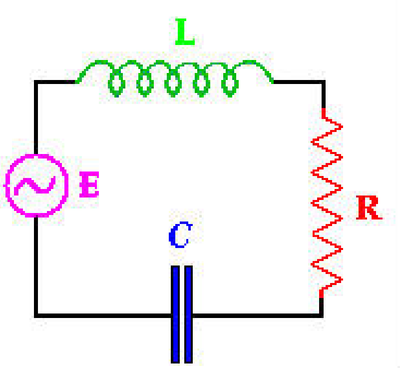
Electric Circuit - no external voltage source

$$\left(LD^2 + RD + \frac{1}{C}\right)Q(t) = 0$$



Add a voltage source to the circuit:

$$\left(LD^2 + RD + \frac{1}{C}\right) Q(t) = \mathcal{E}(t)$$



# Linearity

$$D(c_1 f(x) + c_2 g(x)) = c_1 Df(x) + c_2 Dg(x)$$
$$D^2(c_1 f(x) + c_2 g(x)) = c_1 D^2 f(x) + c_2 D^2 g(x)$$
$$D^3(c_1 f(x) + c_2 g(x)) = c_1 D^3 f(x) + c_2 D^3 g(x)$$
$$\vdots$$
$$D^k(c_1 f(x) + c_2 g(x)) = c_1 D^k f(x) + c_2 D^k g(x)$$

$$
\begin{aligned}
P(D)(c_1 f(x) + c_2 g(x)) &= \sum_{k=0}^{n} a_k D^k (c_1 f(x) + c_2 g(x)) \\
&= \sum_{k=0}^{n} a_k (c_1 D^k f(x) + c_2 D^k g(x)) \\
&= c_1 \sum_{k=0}^{n} a_k D^k f(x) + c_2 \sum_{k=0}^{n} a_k D^k g(x) \\
&= c_1 P(D) f(x) + c_2 P(D) g(x)
\end{aligned}
$$

If $P(D)y_1 = 0$ and $P(D)y_2 = 0$ then any linear combination of $y_1$ and $y_2$ will also solve $P(D)y = 0$

$$
\begin{aligned}
P(D)(c_1y_1 + c_2y_2) &= c_1P(D)y_1 + c_2P(D)y_2 \\
&= c_1 \cdot 0 + c_2 \cdot 0 \\
&= 0
\end{aligned}
$$

Example:

$$(D^2 - 4)y = 0$$

$e^{2x}$ and $e^{-2x}$ are solutions.

A more general solution would be:

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

$$(D^2 - 4)y = 2x$$

If $y_1$ and $y_2$ are solutions of this equation, will any linear combination also be a solution?

$$(D^2 - 4)y = 2x$$

$$(D^2 - 4)y_1 = 2x \qquad (D^2 - 4)y_2 = 2x$$

$$\begin{aligned}(D^2 - 4)(c_1 y_1 + c_2 y_2) &= c_1(D^2 - 4)y_1 + c_2(D^2 - 4)y_2 \\ &= c_1 \cdot 2x + c_2 \cdot 2x \\ &= (c_1 + c_2) \cdot 2x\end{aligned}$$

$$(D^2 - 4)y = 2x$$

Will there be any solutions of the form $y = e^{rx}$ ?

$$(D^2 - 4)y = 2x$$

Will there be any solutions of the form $y = e^{rx}$ ?

$$(D^2 - 4)\left(e^{rx}\right) = 2x$$

$$r^2 e^{rx} - 4e^{rx} = 2x$$

$$r^2 - 4 = 2xe^{-rx}$$

**Contradiction!**

Let's compare the solution of

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{0}}$$

to the solution of

$$\mathbf{A}\vec{\mathbf{X}} = \vec{\mathbf{F}}$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x - 2y \\ -2x + 4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Comparing coordinates: $x = 2y$

$$\vec{\mathbf{X}}_h = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x - 2y \\ -2x + 4y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Comparing coordinates: $\quad x = 2y + 1$

$$\vec{\mathbf{X}} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y + 1 \\ y \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x - 2y \\ -2x + 4y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\vec{\mathbf{X}} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y + 1 \\ y \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{\mathbf{X}} = \vec{\mathbf{X}}_h + \vec{\mathbf{X}}_p$$

$$P(D)y = 0$$

Suppose the general homogeneous solution is a linear combination of:

$$y_1,\ y_2,\ \cdots, y_n$$

What can we say about the solution of:

$$P(D)y = f(x)$$

Suppose we have two functions $y_p$ and $y_q$ that solve $P(D)y = f(x)$

$$P(D)y_p = f(x) \qquad P(D)y_q = f(x)$$

$$P(D)(y_q - y_p) = P(D)y_q - P(D)y_p = f(x) - f(x) = 0$$

$y_q - y_p$ solves $P(D)y = 0$

Since $y_q - y_p$ solves $P(D)y = 0$, it must be a linear combination of $y_1$, $y_2$, $\cdots$, $y_n$

$$y_q - y_p = c_1 y_1 + c_2 y_2 + \cdots c_n y_n$$

$$y_q = c_1 y_1 + c_2 y_2 + \cdots c_n y_n + y_p$$

$$y_q = y_h + y_p$$