Differential Equations
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Today's Topic : Method of Undetermined Coefficients

Theorem:
Suppose y_h is the general solution of $P(D)y = 0$
Let y_p be *any* solution of $P(D)y = f(x)$

The general solution of $P(D)y = f(x)$ is:

$$y = y_p + y_h$$

$$D^2y = 12x + 4$$

We know that the general solution is $y = y_p + y_h$
Start with the homogeneous equation $D^2y = 0$

$$D^2 y = 0$$

$$Dy = \int 0 \, dx = 0 + \text{const}$$

$$y = \int (\text{const}) \, dx = (\text{const})x + \text{const}$$

$$D^2 y = 0$$

$$Dy = \int 0 \, dx = 0 + \text{const}$$

$$y = \int (\text{const}) \, dx = (\text{const})x + \text{const}$$

$$y_h = c_1 + c_2 x$$

$$D^2y = 12x + 4$$

$$Dy = \int (12x + 4)\, dx = 6x^2 + 4x + C_2$$

$$y = \int (6x^2 + 4x + C_2)\, dx = 2x^3 + 2x^2 + C_2x + C_1$$

Note that $y_h = C_1 + C_2x$ and $y_p = 2x^2 + 2x^3$ so the solution is now in the form:

$$y = y_p + y_h$$

Alternative approach:

$$D^2 y = 12x + 4$$

$$D^2 D^2 y = D^2(12x + 4)$$

$$D^4 y = 0$$

D^2 is an *annihilator* of $12x + 4$
Any solution of $D^2 y = 12x + 4$ also solves $D^4 y = 0$

$$D^2y = 12x + 4$$

$$D^4y = 0$$

$$y = c_1 + c_2x + c_3x^2 + c_4x^3$$

$$D^2 y = 12x + 4$$

$$D^4 y = 0$$

$$y = \boxed{c_1 + c_2 x} + \boxed{c_3 x^2 + c_4 x^3}$$

y_h $\qquad\qquad$ y_p

(general form)

Equation 1 $$D^2 y = 12x + 4$$

Equation 2 $$D^4 y = 0$$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

The actual solution is:

$$y = c_1 + c_2 x + 2x^2 + 2x^3$$

Example - Method of Undetermined Coefficients
Solve the equation:

$$(D^2 + 4)y = 2 + 4x + 12x^2$$

The solution has the form $y = y_p + y_h$
Find y_h by solving $(D^2 + 4)y = 0$

$$(D^2 + 4) = 0$$

Substitute e^{rx}

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

The solution is a linear combination of e^{2ix} and e^{-2ix}

Using Euler's formula, we can write this in terms of $\cos 2x$ and $\sin 2x$

$$(D^2 + 4)y = 2 + 4x + 12x^2$$

The solution has the form:

$$y = y_p + y_h = y_p + c_1 \cos 2x + c_2 \sin 2x$$

$$(D^2+4)y = 2+4x+12x^2$$
$$D^3(D^2+4)y = D^3(2+4x+12x^2)$$
$$D^3(D^2+4)y = 0$$

To solve, substitute e^{rx}

$$r^3(r^2+4) = 0$$

$r = \pm 2i$ and $r = 0$ (repeated)

$$\text{Solutions:} \quad \cos 2x,\ \sin 2x,\ e^{0x},\ xe^{0x},\ x^2e^{0x}$$

Equation 1 $\qquad (D^2 + 4)y = 2 + 4x + 12x^2$

Equation 2 $\qquad D^3(D^2 + 4)y = 0$

$$y = a_1 \cos 2x + a_2 \sin 2x + a_3 e^{0x} + a_4 x e^{0x} + a_5 x^2 e^{0x}$$
$$= \underbrace{a_1 \cos 2x + a_2 \sin 2x}_{y_h} + \underbrace{b_1 + b_2 x + b_3 x^2}_{y_p}$$

y_h $\qquad$ y_p

(General form)

Solve for b_1, b_2 and b_3 so that:

$$(D^2 + 4)(b_1 + b_2 x + b_3 x^2) = 2 + 4x + 12x^2$$

Solve for b_1, b_2 and b_3 so that:

$$(D^2 + 4)(b_1 + b_2 x + b_3 x^2) = 2 + 4x + 12x^2$$

$$y_p = b_1 + b_2 x + b_3 x^2$$

$$D^2 y_p = 2b_3$$

Solve for b_1, b_2 and b_3 so that:

$$(D^2+4)(b_1+b_2x+b_3x^2)=2+4x+12x^2$$

$$4y_p=4b_1+4b_2x+4b_3x^2$$

$$D^2y_p=2b_3$$

$$(D^2+4)y_p=D^2y_p+4y_p=4b_1+2b_3+4b_2x+4b_3x^2$$

Solve for b_1, b_2 and b_3 so that:

$$(D^2+4)(b_1+b_2x+b_3x^2) = 2+4x+12x^2$$

$$4y_p = 4b_1+4b_2x+4b_3x^2$$

$$D^2y_p = 2b_3$$

$$(D^2+4)y_p = D^2y_p+4y_p = 4b_1+2b_3+4b_2x+4b_3x^2$$

$$4b_3 = 12 \text{ so } b_3 = 3$$

$$4b_2 = 4 \text{ so } b_2 = 1$$

$$4b_1 + 2b_3 = 2 \text{ so } b_1 = \frac{1}{2}(1 - b_3) = -1$$

$$y_p = b_1 + b_2 x + b_3 x^2 = -1 + x + 3x^2$$

$$(D^2 + 4)y = 2 + 4x + 12x^2$$

$$y = a_1 \cos 2x + a_2 \sin 2x - 1 + x + 3x^2$$

Solve the equation:

$$(D^2 + 4)y = e^{2x}$$

$$y = y_p + y_h$$

We already know that $y_h = a_1 \cos 2x + a_2 \sin 2x$

$$D\left(e^{2x}\right) = 2e^{2x}$$

$$D\left(e^{2x}\right) = 2e^{2x}$$

$$D\left(e^{2x}\right) - 2e^{2x} = 0$$

$$(D-2)\left(e^{2x}\right) = 0$$

The operator $D-2$ annihilates e^{2x}
More generally, $D-r$ annihilates e^{rx}

$$(D^2 + 4)y = e^{2x}$$

$$(D - 2)(D^2 + 4)y = (D - 2)\left(e^{2x}\right)$$

$$(D - 2)(D^2 + 4)y = 0$$

$$(D-2)(D^2+4)y=0$$

Solution is a linear combination of $\cos 2x$, $\sin 2x$, e^{2x}

$$y=a_1\cos 2x+a_2\sin 2x+be^{2x}=y_h+y_p$$

Solve for b so that $(D^2+4)y_p=e^{2x}$

Solve for b so that $(D^2 + 4)(be^{2x}) = e^{2x}$

$$4be^{2x} + 4be^{2x} = e^{2x}$$

$$8be^{2x} = e^{2x}$$

$$b = \frac{1}{8}$$

$$y_p = be^{2x} = \frac{1}{8}e^{2x}$$

$$(D^2+4)y = e^{2x}$$

$$y = \frac{1}{8}e^{2x} + a_1 \cos 2x + a_2 \sin 2x$$

Example: Find the general solution of:

$$(D^2 + 2D + 1)y = x^2 + e^{2x}$$

Start by finding y_h

To find y_h, solve:

$$(D^2 + 2D + 1)y = 0$$

$$(D + 1)^2 y = 0$$

$$y = a_1 e^{-x} + a_2 x e^{-x}$$

Example: Find the general solution of:

$$(D^2 + 2D + 1)y = x^2 + e^{2x}$$

$$y = y_p + a_1 e^{-x} + a_2 x e^{-x}$$

$$(D^2 + 2D + 1)y = x^2 + e^{2x}$$

Could try guessing the form of y_p

$$y_p = b_1 + b_2 x + b_3 x^2 + c e^{2x}$$

$$(D^2 + 2D + 1)y = x^2 + e^{2x}$$

D^3 is the annihilator of x^2
$(D - 2)$ is the annihilator of e^{2x}

$$\begin{aligned}
D^3(D-2)\left(x^2+e^{2x}\right) &= D^3(D-2)\left(x^2\right)+D^3(D-2)\left(e^{2x}\right) \\
&= (D-2)D^3\left(x^2\right)+D^3(D-2)\left(e^{2x}\right) \\
&= (D-2)(0)+D^3(0) \\
&= 0
\end{aligned}$$

$$(D^2 + 2D + 1)y = x^2 + e^{2x}$$

$$D^3(D - 2)(D^2 + 2D + 1)y = D^3(D - 2)\left(x^2 + e^{2x}\right)$$

$$D^3(D - 2)(D^2 + 2D + 1)y = 0$$

$$D^3(D-2)(D^2+2D+1)y = 0$$

Substitute e^{rx}

$$r^3(r-2)(r+1)^2 = 0$$

Solutions: $r = 0$ (repeated) $\quad r = 2 \quad r = -1$ (repeated)

$$y = a_1e^{-x} + a_2xe^{-x} + b_1e^{0x} + b_2xe^{0x} + b_3x^2e^{0x} + ce^{2x}$$

$$y_p = b_1 + b_2x + b_3x^2 + ce^{2x}$$

$$(D^2 + 2D + 1)y = x^2 + e^{2x}$$

$$y_p = b_1 + b_2 x + b_3 x^2 + ce^{2x}$$

Solve for b_1, b_2, b_3 and c

$$2Dy_p = 2b_2 + 4b_3 x + +4ce^{2x}$$

$$D^2 y_p = 2b_3 + 4ce^{2x}$$

$$(D^2 + 2D + 1)y = x^2 + e^{2x}$$

$$y_p = b_1 + b_2 x + b_3 x^2 + ce^{2x}$$

$$2Dy_p = 2b_2 + 4b_3 x + 4ce^{2x}$$

$$D^2 y_p = 2b_3 + 4ce^{2x}$$

$$(D^2 + 2D + 1)y_p = b_1 + 2b_2 + 2b_3 + (b_2 + 4b_3)x + b_3 x^2 + 9ce^{2x}$$

$$(D^2 + 2D + 1)y = x^2 + e^{2x}$$

$$y_p = b_1 + b_2 x + b_3 x^2 + ce^{2x}$$

$$2Dy_p = 2b_2 + 4b_3 x + 4ce^{2x}$$

$$D^2 y_p = 2b_3 + 4ce^{2x}$$

$$(D^2 + 2D + 1)y_p = b_1 + 2b_2 + 2b_3 + (b_2 + 4b_3)x + b_3 x^2 + 9ce^{2x}$$

0 0 1 1

$$9c = 1 \text{ so } c = \frac{1}{9}$$

$$b_3 = 1 \text{ so } b_3 = 1$$

$$b_2 + 4b_3 = 0 \text{ so } b_2 = -4$$

$$b_1 + 2b_2 + 2b_3 = 0 \text{ so } b_1 = 6$$

$$y_p = b_1 + b_2 x + b_3 x^2 + ce^{2x} = 6 - 4x + x^2 + \frac{1}{9}e^{2x}$$

$$(D^2 + 2D + 1)y = x^2 + e^{2x}$$

$$y = 6 - 4x + x^2 + \frac{1}{9}e^{2x} + a_1 e^{-x} + a_2 x e^{-x}$$