Differential Equations
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Today's Topic : Annihilators

$$(D^2 + 4)y = e^{2x}$$

$$(D - 2)(D^2 + 4)y = (D - 2)\left(e^{2x}\right)$$

$$(D - 2)(D^2 + 4)y = 0$$

An *annihilator* of $f(x)$ is an operator $A(D)$ such that

$$A(D)f(x) = 0$$

Annihilators:

$$D^2(4x+5)=0$$

$$(D-2)\left(e^{2x}\right)=0$$

$$P(D)y = f(x)$$

$$A(D)P(D)y = A(D)f(x)$$

$$A(D)P(D)y = 0$$

Function	Annihilator
x^n	D^{n+1}
e^{rx}	$D - r$

Exponential Shift Theorem:

$$P(D)\left(e^{rx}f(x)\right) = e^{rx}P(D+r)f(x)$$

$$P(D)\left(e^{rx} f(x)\right) = e^{rx} P(D+r) f(x)$$

Examples:

$$(D-r)^2\left(x\, e^{rx}\right) = e^{rx} D^2(x) = 0$$
$$(D-r)^3\left(x^2 e^{rx}\right) = e^{rx} D^3(x^2) = 0$$
$$(D-r)^4\left(x^3 e^{rx}\right) = e^{rx} D^4(x^3) = 0$$

Function	Annihilator
x^n	D^{n+1}
e^{rx}	$D - r$
$x^n e^{rx}$	$(D - r)^{n+1}$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = xe^{2x}$$

Equivalently,

$$(D^2 - 2D)y = xe^{2x}$$

Start by finding the homogeneous solution y_h

Homogeneous equation:

$$(D^2 - 2D)y = 0$$

Substitute e^{rx}

$$r^2 - 2r = 0$$

$$r(r - 2) = 0$$

Solutions are $r = 0$ and $r = 2$

$$y_h = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$$

$$(D^2 - 2D)y = xe^{2x}$$

$$y = y_p + y_h = y_p + c_1 + c_2e^{2x}$$

$$(D^2 - 2D)y = xe^{2x}$$

$$(D-2)^2(D^2 - 2D)y = (D-2)^2(xe^{2x})$$

$$(D-2)^2(D^2 - 2D)y = 0$$

$$(D-2)^2(D^2-2D)y=0$$

$$(D-2)^2(D-2)Dy=0$$

$$(D-2)^3Dy=0$$

Substitute e^{rx}

$$(r-2)^3r=0$$

$$\text{Solutions :} \quad r=2 \text{ (repeated)} \qquad r=0$$

$$y=c_1e^{0x}+c_2e^{2x}+b_1xe^{2x}+b_2x^2e^{2x}$$

$$(D-2)^2(D^2-2D)y=0$$

$$(D-2)^2(D-2)Dy=0$$

$$(D-2)^3Dy=0$$

Substitute e^{rx}

$$(r-2)^3r=0$$

$$\text{Solutions :} \quad r=2 \text{ (repeated)} \qquad r=0$$

$$y = \boxed{c_1e^{0x}+c_2e^{2x}} + \boxed{b_1xe^{2x}+b_2x^2e^{2x}}$$

y_h y_p

(general form)

Solve for b_1 and b_2 so that y_p solves $(D^2 - 2D)y_p = xe^{2x}$

$$(D^2 - 2D)\left(b_1 x e^{2x} + b_2 x^2 e^{2x}\right) = xe^{2x}$$

$$\left(D^2 - 2D\right)\left(b_1 x e^{2x} + b_2 x^2 e^{2x}\right) = x e^{2x}$$

$$e^{2x}\left((D+2)^2 - 2(D+2)\right)\left(b_1 x + b_2 x^2\right) = x e^{2x}$$

$$e^{2x}\left(D^2 + 2D\right)\left(b_1 x + b_2 x^2\right) = x e^{2x}$$

$$e^{2x} \cdot (2b_1 + 2b_2 + 4b_2 x) = x e^{2x}$$

$$2b_1 + 2b_2 + 4b_2 x = x$$

$$2b_1 + 2b_2 = 0 \qquad 4b_2 = 1$$

$$2b_1 + 2b_2 = 0 \qquad 4b_2 = 1$$

$$b_1 = -b_2 \qquad b_2 = \frac{1}{4}$$

$$y_p = b_1 x e^{2x} + b_2 x^2 e^{2x} = -\frac{1}{4} x e^{2x} + \frac{1}{4} x^2 e^{2x}$$

$$(D^2 - 2D)y = xe^{2x}$$

$$y = c_1 + c_2 e^{2x} - \frac{1}{4}xe^{2x} + \frac{1}{4}x^2 e^{2x}$$

Example: Find the solution of:

$$\frac{d^2y}{dx^2} - y = \sin x \qquad \text{where } y(0) = y'(0) = 0$$

Start by finding y_h

$$\frac{d^2y}{dx^2} - y = 0$$

Substitute e^{rx}

$$r^2 - 1 = 0$$

$$r = \pm 1$$

$$y_h = c_1 e^x + c_2 e^{-x}$$

$$\frac{d^2y}{dx^2} - y = \sin x \qquad \text{where } y(0) = y'(0) = 0$$

$$y = y_p + c_1 e^x + c_2 e^{-x}$$

What is the annihilator of $\sin x$?

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

Solve for $\cos x$ and $\sin x$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

Since $(D - i)(e^{ix}) = 0$ and $(D + i)e^{-ix} = 0$, $(D - i)(D + i)$ will annihilate any linear combination of e^{ix} and e^{-ix}

$$(D-i)(D+i) = D^2 - i^2 = D^2 - (-1) = D^2 + 1$$

$$(D^2+1)(a\cos x + b\sin x) = 0$$

More generally,

$$(D^2+\omega^2)(a\cos \omega x + b\sin \omega x) = 0$$

$$\frac{d^2y}{dx^2} - y = \sin x \qquad \text{where } y(0) = y'(0) = 0$$

$$(D^2 - 1)y = \sin x$$

$$(D^2 + 1)(D^2 - 1)y = (D^2 + 1)\sin x$$

$$(D^2 + 1)(D^2 - 1)y = 0$$

Substitute e^{rx} and solve for r

$$r = \pm 1 \qquad r = \pm i$$

$$y = c_1 e^x + c_2 e^{-x} + a\cos x + b\sin x$$

Solve for a and b so that $y_p = a\cos x + b\sin x$ solves

$$(D^2+1)y_p = \sin x$$

$$(D^2-1)(a\cos x + b\sin x) = \sin x$$

$$(-a\cos x - b\sin x) - (a\cos x + b\sin x) = \sin x$$

$$-2a\cos x - 2b\sin x = 0\cos x + 1\sin x$$

$$a = 0 \qquad b = -\frac{1}{2}$$

$$y_p = -\frac{1}{2}\sin x$$

$$\frac{d^2y}{dx^2} - y = \sin x \qquad \text{where } y(0) = y'(0) = 0$$

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}\sin x$$

$y(0) = 0$ implies that $0 = c_1 + c_2$ so $c_2 = -c_1$

$$y = c_1 e^x - c_1 e^{-x} - \frac{1}{2}\sin x$$

$$y = c_1 e^x - c_1 e^{-x} - \frac{1}{2}\sin x$$

$$y' = c_1 e^x + c_1 e^{-x} - \frac{1}{2}\cos x$$

$$0 = y'(0) = c_1 + c_1 - \frac{1}{2}$$

$$c_1 = \frac{1}{4}$$

$$y = \frac{1}{4}e^x - \frac{1}{4}e^{-x} - \frac{1}{2}\sin x$$

$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

Hyperbolic sine and hyperbolic cosine:

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$
\begin{aligned}
y &= \frac{1}{4}e^{x} - \frac{1}{4}e^{-x} - \frac{1}{2}\sin x \\
&= \frac{1}{2} \cdot \frac{1}{2}\left(e^{x} - e^{-x}\right) - \frac{1}{2}\sin x \\
&= \frac{1}{2}\sinh x - \frac{1}{2}\sin x
\end{aligned}
$$