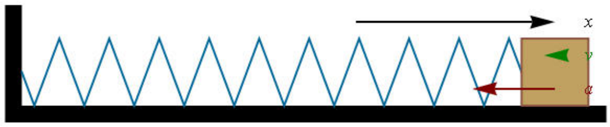
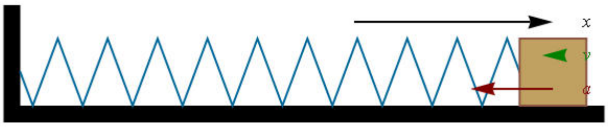
Differential Equations
Dr. E. Jacobs

Today's Topic : $my'' + \beta y' + ky = F(t)$

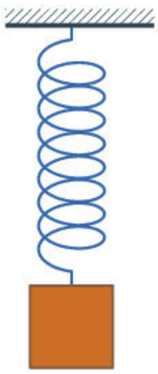
$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = F(t)$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = mg$$



Start with the homogeneous equation

$$(mD^2 + \beta D + k)y = 0$$

$$\text{Let} \quad \omega = \frac{\sqrt{4mk - \beta^2}}{2m}$$

$$y(t) = c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t$$

$$(mD^2 + \beta D + k)y = mg$$

$$y = y_p + y_h = y_p + c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t$$

$$(mD^2 + \beta D + k)y = mg$$

$$D(mD^2 + \beta D + k)y = D(mg)$$

$$D(mD^2 + \beta D + k)y = 0$$

Substitute e^{rt} to obtain $r(mr^2 + \beta r + k) = 0$
The solution of $mr^2 + \beta r + k = 0$ leads to y_h
The solution $r = 0$ gives us the general form of y_p

$$y_p = ce^{0t} = c$$

Substitute $y_p = c$

$$(mD^2 + \beta D + k)y = mg$$

$$(mD^2 + \beta D + k)(c) = mg$$

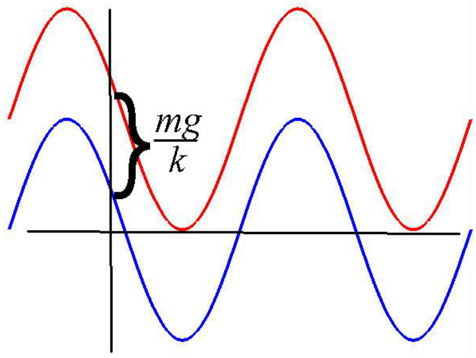
$$kc = mg$$

$$c = \frac{mg}{k}$$

$$(mD^2 + \beta D + k)y = mg$$

$$y(t) = \frac{mg}{k} + c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t$$

$$y(t) = \frac{mg}{k} + c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t$$



If the mass at the end of the spring is displaced x meters, the force of the spring is:

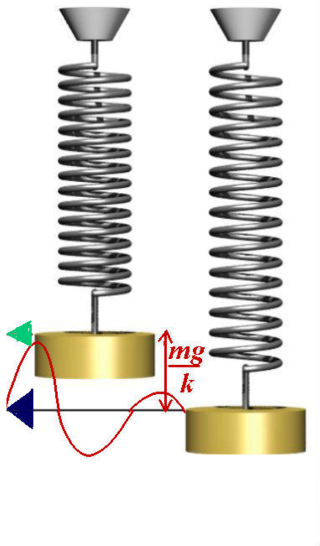
$$F_s = kx$$

Force of gravity is mg. These forces balance when:

$$kx = mg$$

$$x = \frac{mg}{k}$$

$$y(t) = \frac{mg}{k} + c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t$$

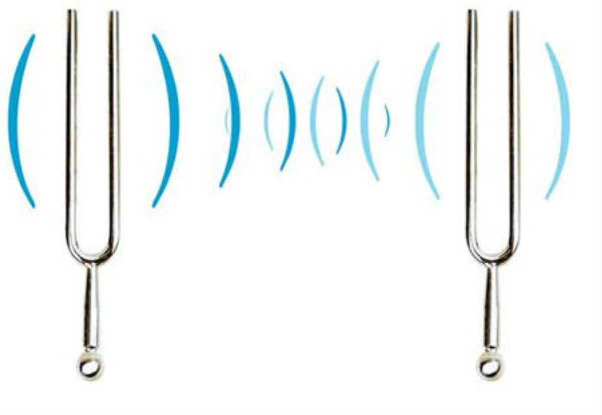


Periodic Forcing Function

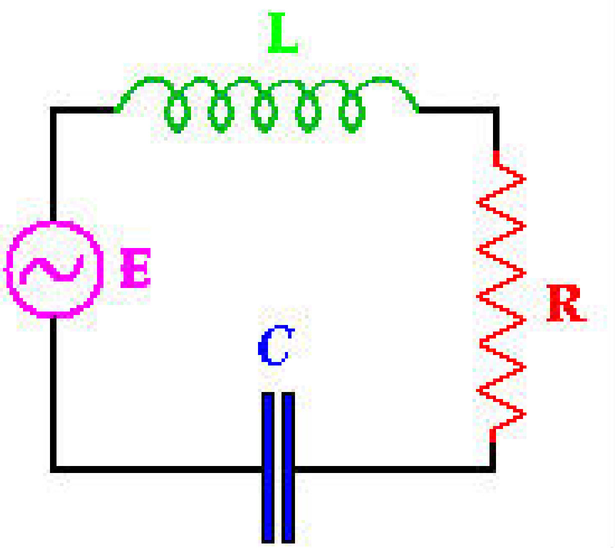
$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = F(t)$$

$$\text{Let} \quad F(t) = \cos\gamma t$$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = \cos\gamma t$$



$$L\frac{d^2Q}{dt^2} + RQ + \frac{1}{C}Q(t) = \cos\gamma t$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = \cos\gamma t$$

Let $m = 1$, $\beta = 2$, $k = 9$ and $\gamma = 3$

$$(D^2 + 2D + 9)y = \cos 3t$$

$$y = y_p + y_h$$

Start by solving $(D^2 + 2D + 9)y = 0$ to find y_h

$$(D^2 + 2D + 9)y = 0$$

Substitute e^{rt}

$$r^2 + 2r + 9 = 0$$

$$r^2 + 2r + 1 = -8$$

$$(r + 1)^2 = -8$$

$$r = -1 \pm \sqrt{-8} = -1 \pm 2\sqrt{2}i$$

If $r = -1 \pm 2\sqrt{2}i$ then the solution is a linear combination of $e^{(-1+2\sqrt{2}i)t}$ and $e^{(-1-2\sqrt{2}i)t}$

Apply Euler's formula and you will obtain a linear combination of $e^{-t} \cos 2\sqrt{2}t$ and $e^{-t} \sin 2\sqrt{2}t$

$$(D^2 + 2D + 9)y = \cos 3t$$

$$y = y_p + y_h = y_p + c_1 e^{-t} \cos 2\sqrt{2}t + c_2 e^{-t} \sin 2\sqrt{2}t$$

$$(D^2 + 2D + 9)y = \cos 3t$$

$$(D^2 + 9)(D^2 + 2D + 9)y = (D^2 + 9)(\cos 3t)$$

$$(D^2 + 9)(D^2 + 2D + 9)y = 0$$

Substitute e^{rt}

$$(r^2 + 9)(r^2 + 2r + 9) = 0$$

The solution of $r^2 + 2r + 9 = 0$ leads to y_h

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

y_p is a linear combination of e^{3it} and e^{-3it}
We can use Euler's formula to write this as a linear combination of $\cos 3t$ and $\sin 3t$

$$y_p = A\cos 3t + B\sin 3t$$

This is the *general form* of y_p

Solve for A and B so that $y_p = A\cos 3t + B\sin 3t$ solves $(D^2 + 2D + 9)y = \cos 3t$

$$y_p = A\cos 3t + B\sin 3t$$

$$Dy_p = 3B\cos 3t - 3A\sin 3t$$

$$D^2y_p = -9A\cos 3t - 9B\sin 3t$$

Solve for A and B so that $y_p = A\cos 3t + B\sin 3t$ solves $(D^2 + 2D + 9)y = \cos 3t$

$$9y_p = 9A\cos 3t + 9B\sin 3t$$

$$2Dy_p = 6B\cos 3t - 6A\sin 3t$$

$$D^2 y_p = -9A\cos 3t - 9B\sin 3t$$

Add these up

$$(D^2 + 2D + 9)y_p = 6B\cos 3t - 6A\sin 3t$$

$$(D^2 + 2D + 9)y_p = 6B\cos 3t - 6A\sin 3t$$

We want to solve $(D^2 + 2D + 9)y = \cos 3t$, so

$$6B = 1 \quad \text{and} \quad -6A = 0$$

$$B = \frac{1}{6} \quad \text{and} \quad A = 0$$

$$y_p = \frac{1}{6}\sin 3t$$

$$(D^2 + 2D + 9)y = \cos 3t$$

$$y = \frac{1}{6} \sin 3t + c_1 e^{-t} \cos 2\sqrt{2}t + c_2 e^{-t} \sin 2\sqrt{2}t$$

$$(D^2 + 2D + 9)y = \cos 3t$$

$$y = \frac{1}{6}\sin 3t + c_1 e^{-t}\cos 2\sqrt{2}t + c_2 e^{-t}\sin 2\sqrt{2}t$$

Add on initial conditions : $y(0) = \frac{1}{2}$ and $y'(0) = 0$

$$y = \underbrace{\boxed{\frac{1}{6}\sin 3t}}_{y_p} + \underbrace{\boxed{\frac{1}{2}e^{-t}\cos 2\sqrt{2}t}}_{y_h}$$

$$(D^2 + 2D + 9)y = \cos 3t$$

$$y = \frac{1}{6}\sin 3t + c_1 e^{-t}\cos 2\sqrt{2}t + c_2 e^{-t}\sin 2\sqrt{2}t$$

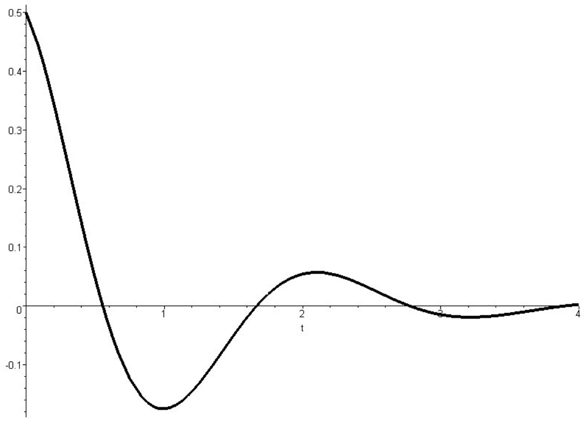
Add on initial conditions : $y(0) = \frac{1}{2}$ and $y'(0) = 0$

$$y = \boxed{\frac{1}{6}\sin 3t} + \boxed{\frac{1}{2}e^{-t}\cos 2\sqrt{2}t}$$

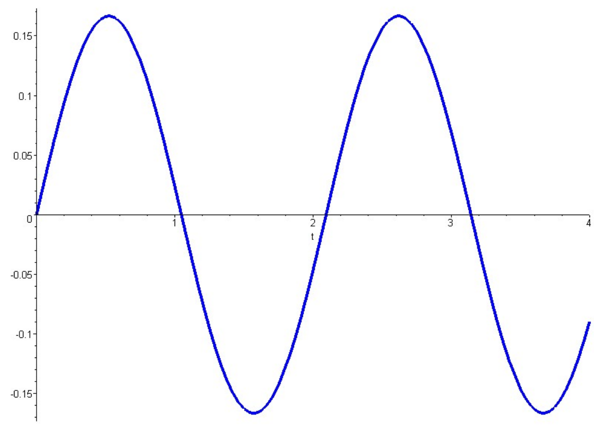
steady-state part ***transient part***

$$y_h = \frac{1}{2}e^{-t}\cos 2\sqrt{2}t$$

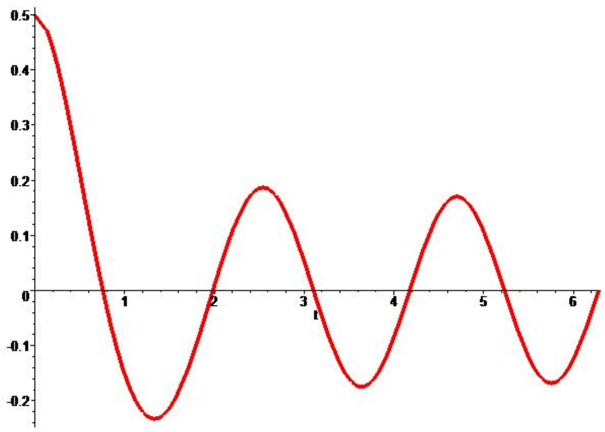
$$\lim_{t\to\infty}\frac{1}{2}e^{-t}\cos 2\sqrt{2}t = 0$$



$$y_p = \frac{1}{6}\sin 3t$$



$$y = \frac{1}{6}\sin 3t + \frac{1}{6}e^{-t}\cos 2\sqrt{2}t$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = \cos\gamma t$$

$$y = y_h + y_p$$

$$y_h = c_1 e^{-\frac{\beta t}{2m}}\cos\omega t + c_2 e^{-\frac{\beta t}{2m}}\sin\omega t$$

$$\text{where } \omega = \frac{\sqrt{4mk - \beta^2}}{2m}$$

$$(mD^2 + \beta D + k)y = \cos \gamma t$$

$$\left(D^2 + \gamma^2\right)\left(mD^2 + \beta D + k\right) y = \left(D^2 + \gamma^2\right)(\cos \gamma t)$$

$$\left(D^2 + \gamma^2\right)\left(mD^2 + \beta D + k\right) y = 0$$

Substitute $y = e^{rt}$

$$\left(D^2+\gamma^2\right)\left(mD^2+\beta D+k\right)y=0$$

Substitute $y=e^{rt}$

$$\left(r^2+\gamma^2\right)\left(mr^2+\beta r+k\right)=0$$

The solution of $r^2+\gamma^2=0$ will give us the particular solution.

$$y_p=A\cos\gamma t+B\sin\gamma t$$

$$y_p = A\cos\gamma t + B\sin\gamma t$$

Solve for A and B so that $(mD^2 + \beta D + k)y_p = \cos\gamma t$

$$y_p = A\cos\gamma t + B\sin\gamma t$$

$$y_p' = B\gamma\cos\gamma t - A\gamma\sin\gamma t$$

$$y_p'' = -A\gamma^2\cos\gamma t - B\gamma^2\sin\gamma t$$

$$y_p = A\cos\gamma t + B\sin\gamma t$$

Solve for A and B so that $(mD^2+\beta D+k)y_p = \cos\gamma t$

$$ky_p = Ak\cos\gamma t + Bk\sin\gamma t$$

$$\beta y_p' = B\beta\gamma\cos\gamma t - A\beta\gamma\sin\gamma t$$

$$my_p'' = -Am\gamma^2\cos\gamma t - Bm\gamma^2\sin\gamma t$$

Add these up

$$
\begin{aligned}
my_p'' + \beta y_p' + ky =& (A(k - m\gamma^2) + B\beta\gamma)\cos\gamma t \\
&+ \left(-\beta\gamma A + B\left(k - m\gamma^2\right)\right)\sin\gamma t
\end{aligned}
$$

We want $my_p'' + \beta y_p' + ky$ to equal $\cos\gamma t$

$$
\begin{aligned}
\left(k - m\gamma^2\right) A \qquad + \qquad \beta\gamma B &= 1 \\
-\beta\gamma A + \left(k - m\gamma^2\right) B &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left(k - m\gamma^2\right) A \qquad + \qquad \beta\gamma B &= 1 \\
-\beta\gamma A + \left(k - m\gamma^2\right) B &= 0
\end{aligned}
$$

$$A = \frac{k - m\gamma^2}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$B = \frac{\beta\gamma}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$A = \frac{k - m\gamma^2}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$B = \frac{\beta\gamma}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$y = c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t + A \cos \gamma t + B \sin \gamma t$$

$$A = \frac{k - m\gamma^2}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$B = \frac{\beta\gamma}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

Please note that this solution assumes

$$\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2 \neq 0$$