Differential Equations

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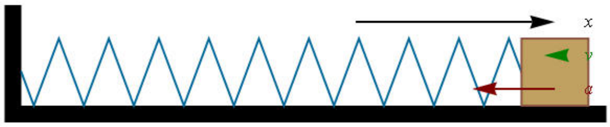
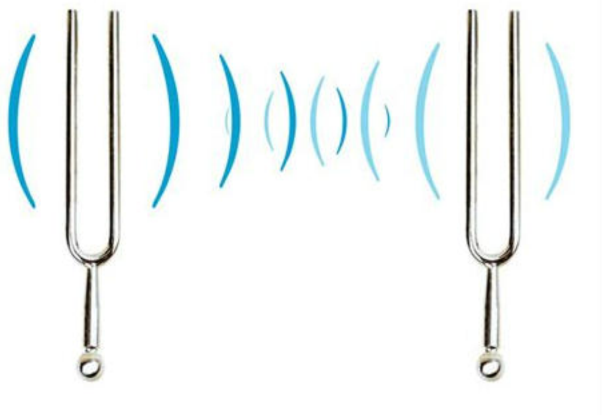
Today's Topic : Resonance

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = F(t)$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = \cos\gamma t$$



$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = \cos\gamma t$$

$$y = y_h + y_p$$

$$y_h = c_1 e^{-\frac{\beta t}{2m}} \cos\omega t + c_2 e^{-\frac{\beta t}{2m}} \sin\omega t$$

$$\text{where } \omega = \frac{\sqrt{4mk - \beta^2}}{2m}$$

$$A = \frac{k - m\gamma^2}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$B = \frac{\beta\gamma}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$y = c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t + A \cos \gamma t + B \sin \gamma t$$

$$A = \frac{k - m\gamma^2}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

$$B = \frac{\beta\gamma}{\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2}$$

Please note that this solution assumes

$$\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2 \neq 0$$

If $\left(k - m\gamma^2\right)^2 + \beta^2\gamma^2 = 0$ then:

$$\left(k - m\gamma^2\right)^2 = 0 \quad \text{and} \quad \beta^2\gamma^2 = 0$$

What would it mean if $\beta = 0$?

$$(mD^2 + \beta D + k)y = \cos \gamma t$$

becomes:

$$(mD^2 + k)y = \cos \gamma t$$

What would it mean if $\beta = 0$?

$$y_h = c_1 e^{-\frac{\beta t}{2m}} \cos \omega t + c_2 e^{-\frac{\beta t}{2m}} \sin \omega t$$

becomes:

$$y_h = c_1 \cos \omega t + c_2 \sin \omega t$$

$$\omega = \frac{\sqrt{4mk - \beta^2}}{2m} = \sqrt{\frac{4mk - \beta^2}{4m^2}} = \sqrt{\frac{4mk}{4m^2}} = \sqrt{\frac{k}{m}}$$

$$\left(k - m\gamma^2\right)^2 = 0 \quad \text{and} \quad \beta^2\gamma^2 = 0$$

What would it mean if $\left(k - m\gamma^2\right)^2 = 0$?

$$m\gamma^2 = k$$

$$\gamma^2 = \frac{k}{m}$$

$$\gamma = \sqrt{\frac{k}{m}} = \omega$$

$$(mD^2 + \beta D + k)y = \cos \gamma t$$

$$(mD^2 + k)y = \cos \omega t$$

$$\left(D^2 + \frac{k}{m}\right) y = \frac{1}{m} \cos \omega t$$

If $\omega = \sqrt{\frac{k}{m}}$ then:

$$\left(D^2 + \omega^2\right) y = \frac{1}{m} \cos \omega t$$

$$\left(D^2 + \omega^2\right) y = \frac{1}{m} \cos \omega t$$

$$y = y_p + c_1 \cos \omega t + c_2 \sin \omega t$$

$$\left(D^2 + \omega^2\right) y = \frac{1}{m} \cos \omega t$$

$$\left(D^2 + \omega^2\right) \left(D^2 + \omega^2\right) y = \left(D^2 + \omega^2\right) \left(\frac{1}{m} \cos \omega t\right)$$

$$\left(D^2 + \omega^2\right) \left(D^2 + \omega^2\right) y = 0$$

$$\left(D^2+\omega^2\right)\left(D^2+\omega^2\right)y=0$$

Substitute e^{rt}

$$(r^2+\omega^2)(r^2+\omega^2)=0$$

$$(r-i\omega)(r+i\omega)(r-i\omega)(r+i\omega)=0$$

$$r=i\omega \qquad r=-i\omega \quad \text{(both roots are repeated)}$$

$$y=a_1e^{i\omega t}+a_2te^{i\omega t}+a_3e^{-i\omega t}+a_4te^{-i\omega t}$$

$$y = a_1 e^{i\omega t} + a_2 t e^{i\omega t} + a_3 e^{-i\omega t} + a_4 t e^{-i\omega t}$$

$e^{i\omega t} = \cos\omega t + i\sin\omega t$ and $e^{-i\omega t} = \cos\omega t - i\sin\omega t$

y can be expressed as a linear combination of:

$$\cos\omega t \qquad \sin\omega t \qquad t\cos\omega t \qquad t\sin\omega t$$

$$y = c_1 \cos \omega t + c_2 \sin \omega t + At \cos \omega t + Bt \sin \omega t$$

$$y_p = At \cos \omega t + Bt \sin \omega t$$

Find A and B by substituting y_p into

$$\left(D^2 + \omega^2\right) y = \frac{1}{m} \cos \omega t$$

$$y_p = At\cos\omega t + Bt\sin\omega t$$

$$y_p' = -\omega At\sin\omega t + A\cos\omega t + \omega Bt\cos\omega t + B\sin\omega t$$

$$y_p = At\cos\omega t + Bt\sin\omega t$$

$$y_p' = -\omega At\sin\omega t + A\cos\omega t + \omega Bt\cos\omega t + B\sin\omega t$$

$$y_p'' = -\omega^2 At\cos\omega t - 2\omega A\sin\omega t - \omega^2 Bt\sin\omega t + 2\omega B\cos\omega t$$

$$\left(D^2 + \omega^2\right) y = \frac{1}{m} \cos \omega t$$

$$\omega^2 y_p = \omega^2 At \cos \omega t + \omega^2 Bt \sin \omega t$$

$$y_p'' = -\omega^2 At \cos \omega t - 2\omega A \sin \omega t - \omega^2 Bt \sin \omega t + 2\omega B \cos \omega t$$

Add to get $D^2 y_p + \omega^2 y_p$

$$(D^2 + \omega^2) y = -2\omega A \sin \omega t + 2\omega B \cos \omega t$$

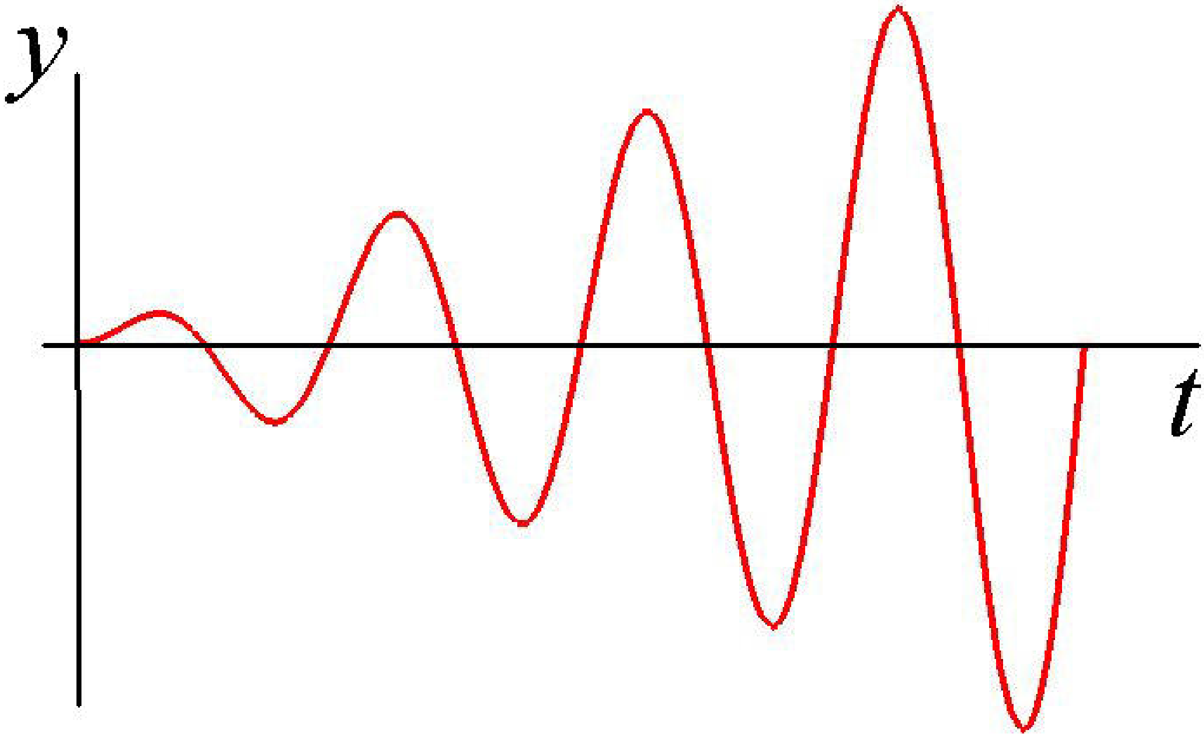
$$(D^2+\omega^2)y=-2\omega A\sin\omega t+2\omega B\cos\omega t$$

We want $\left(D^2+\omega^2\right)y=\frac{1}{m}\cos\omega t$

$$A=0 \qquad B=\frac{1}{2m\omega}$$

$$y_p=At\cos\omega t+Bt\sin\omega t=\frac{1}{2m\omega}t\sin\omega t$$

$$y(t) = c_1 \cos\ \omega t + c_2 \sin\ \omega t + \frac{1}{2m\omega} t \sin\ \omega t$$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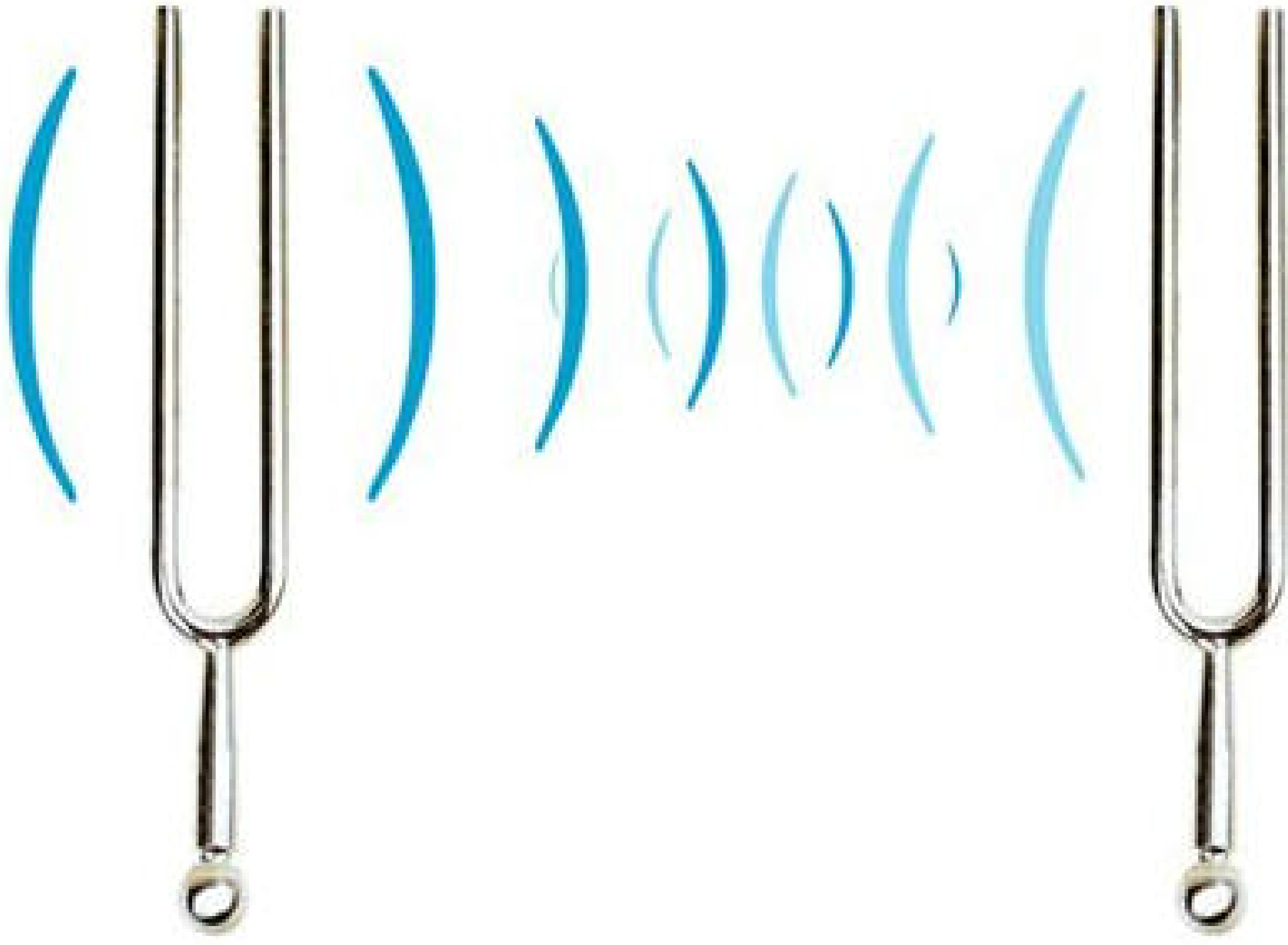

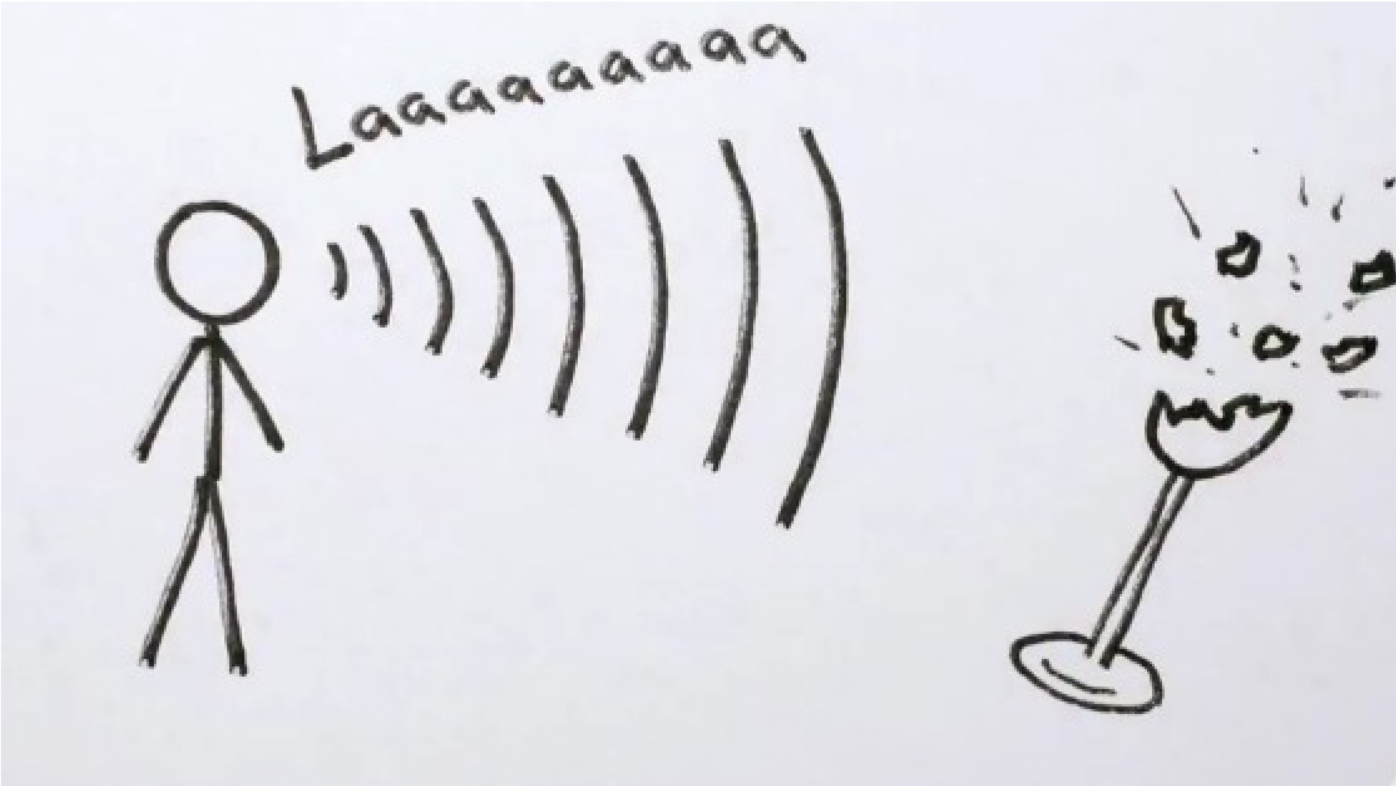
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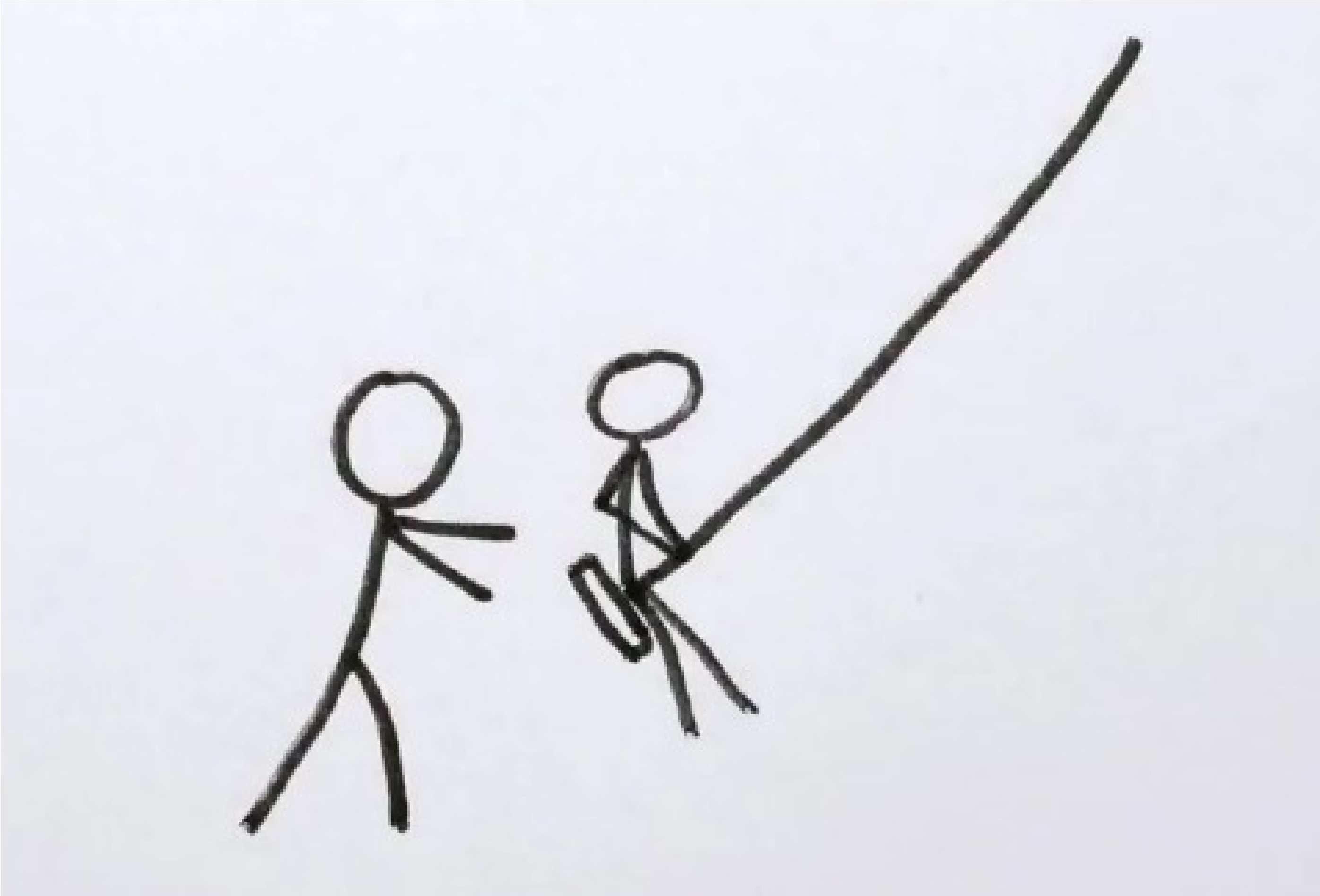

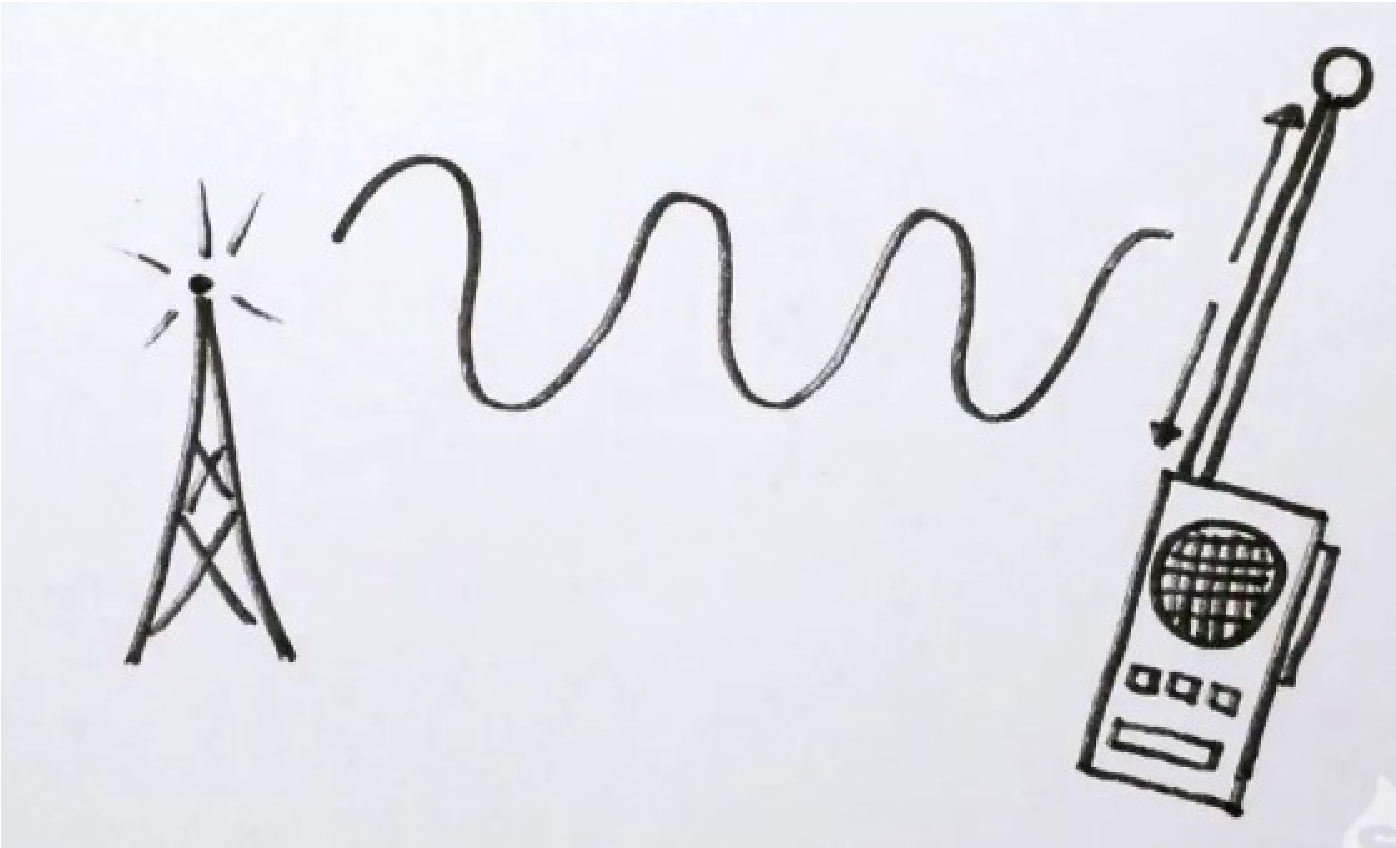

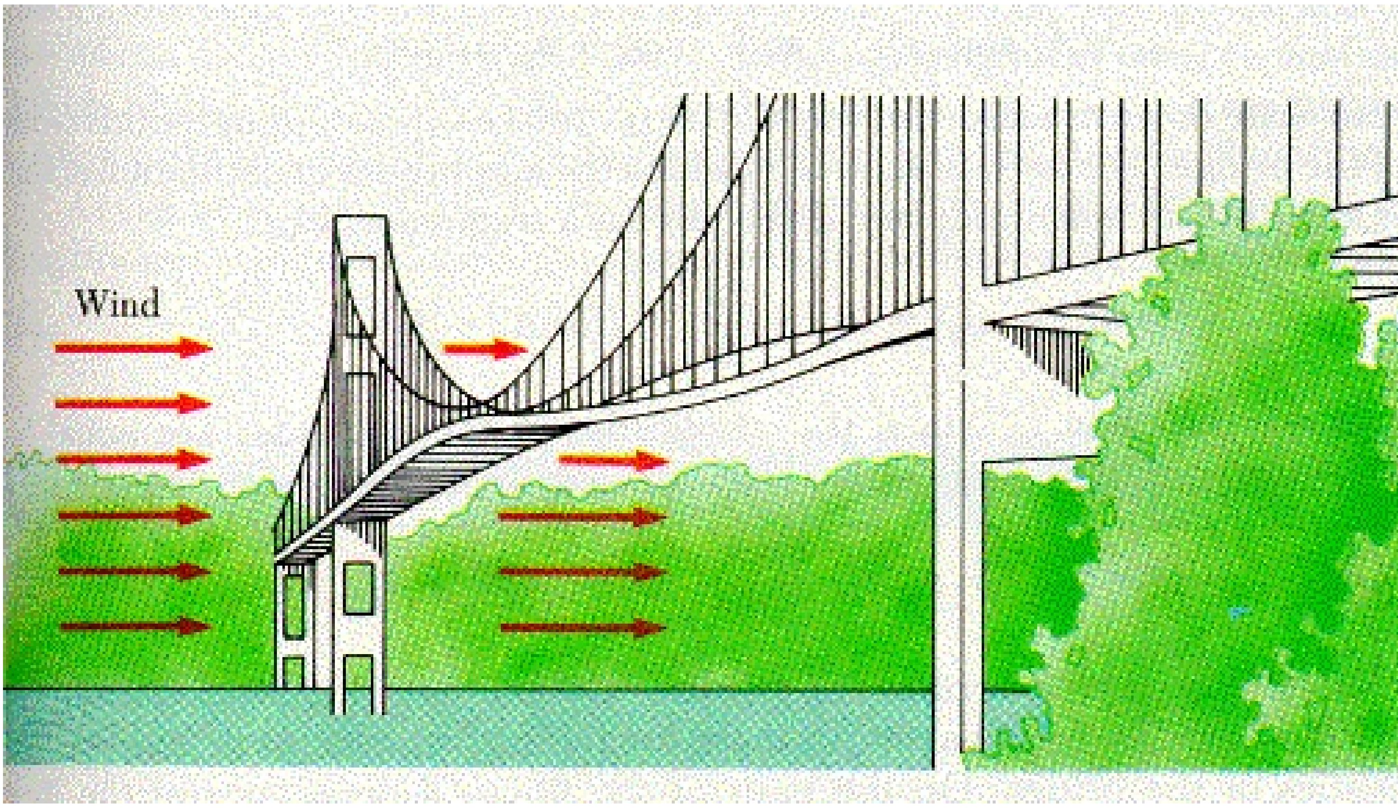
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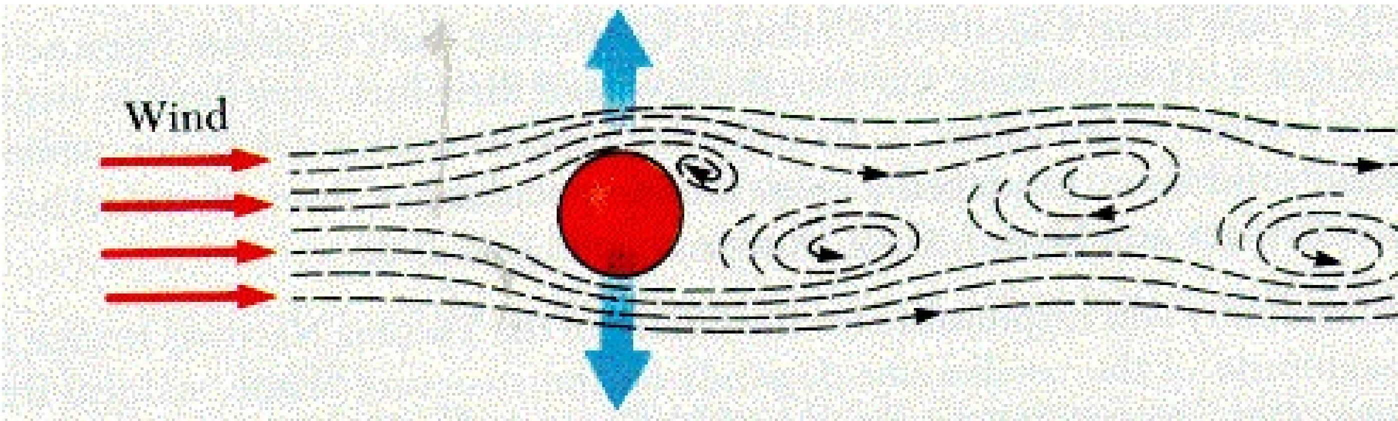
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