Differential Equations
Dr. E. Jacobs

Today's Topic : Variation of Parameters

$$y'' + 4y = x^2$$

$$y_h = c_1 \cos 2x + c_2 \sin 2x$$

$$y = y_h + y_p$$

$$y'' + 4y = x^2$$

$$y_h = c_1 \cos 2x + c_2 \sin 2x$$

$$y = y_h + y_p$$

$$y = c_1 \cos 2x + c_2 \sin 2x + a_0 + a_1 x + a_2 x^2$$

$$y'' + 4y = x^2$$

$$y = c_1 \cos 2x + c_2 \sin 2x + a_0 + a_1 x + a_2 x^2$$

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{8} + \frac{1}{4} x^2$$

$$y'' + 4y = \ln x$$

$$y = c_1 \cos 2x + c_2 \sin 2x + y_p$$

$$y_p = \textbf{???}$$

Or, what if we were trying to solve:

$$y'' + 4y = \sec x$$

Turn to 1st order equations for insight

$$\frac{dy}{dx} + 2y = f(x)$$

$$y_h = ce^{-2x}$$

$$\frac{dy}{dx} + 2y = f(x)$$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x}f(x)$$

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}f(x)$$

$$e^{2x}y = \int e^{2x}f(x)\,dx$$

$$y = \left(\int e^{2x} f(x) \, dx \right) e^{-2x}$$

$$y = v(x)e^{-2x}$$

Compare this with the homogeneous solution:

$$y_h = ce^{-2x}$$

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Suppose the homogeneous solution is

$y = c_1 y_{1h} + c_2 y_{2h}$ where c_1 and c_2 are constants

Can we find functions

$$v_1 = v_1(x) \quad \text{and} \quad v_2 = v_2(x)$$

so that the nonhomogeneous solution is

$$y = v_1 y_{1h} + v_2 y_{2h}$$

$$
\begin{aligned}
ax + by &= f_1 \\
cx + dy &= f_2
\end{aligned}
$$

Solution:

$$
x = \frac{\begin{vmatrix} f_1 & b \\ f_2 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & f_1 \\ c & f_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}
$$

Substitute

$$y = v_1 y_{1h} + v_2 y_{2h}$$

into

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

and solve for v_1 and v_2

$$
\begin{aligned}
y &= v_1 y_{1h} + v_2 y_{2h} \\
y' &= v_1 y'_{1h} + y_{1h} v'_1 + v_2 y'_{2h} + y_{2h} v'_2
\end{aligned}
$$

Switch the two middle terms

$$
\begin{aligned}
y &= v_1 y_{1h} + v_2 y_{2h} \\
y' &= v_1 y'_{1h} + y_{1h} v'_1 + v_2 y'_{2h} + y_{2h} v'_2 \\
&= v_1 y'_{1h} + v_2 y'_{2h} + y_{1h} v'_1 + y_{2h} v'_2
\end{aligned}
$$

$$
\begin{aligned}
y &= v_1 y_{1h} + v_2 y_{2h} \\
y' &= v_1 y'_{1h} + y_{1h} v'_1 + v_2 y'_{2h} + y_{2h} v'_2 \\
&= v_1 y'_{1h} + v_2 y'_{2h} + \boxed{y_{1h} v'_1 + y_{2h} v'_2}
\end{aligned}
$$

Set equal to 0

$$y = v_1 y_{1h} + v_2 y_{2h}$$
$$y' = v_1 y'_{1h} + v_2 y'_{2h}$$

$$y = v_1 y_{1h} + v_2 y_{2h}$$
$$y' = v_1 y'_{1h} + v_2 y'_{2h}$$
$$y'' = v_1 y''_{1h} + y'_{1h} v'_1 + v_2 y''_{2h} + y'_{2h} v'_2$$

Switch the two middle terms

$$\begin{aligned}
y &= v_1 y_{1h} + v_2 y_{2h} \\
y' &= v_1 y'_{1h} + v_2 y'_{2h} \\
y'' &= v_1 y''_{1h} + v_2 y''_{2h} + y'_{1h} v'_1 + y'_{2h} v'_2
\end{aligned}$$

Substitute into $y'' + by' + cy = f(x)$

$$
\begin{aligned}
y &= v_1 y_{1h} + v_2 y_{2h} \\
y' &= v_1 y'_{1h} + v_2 y'_{2h} \\
y'' &= v_1 y''_{1h} + v_2 y''_{2h} + y'_{1h} v'_1 + y'_{2h} v'_2
\end{aligned}
$$

Substitute into $y'' + by' + cy = f(x)$

$$
\begin{aligned}
cy &= cv_1 y_{1h} + cv_2 y_{2h} \\
by' &= bv_1 y'_{1h} + bv_2 y'_{2h} \\
y'' &= \; v_1 y''_{1h} + v_2 y''_{2h} + y'_{1h} v'_1 + y'_{2h} v'_2
\end{aligned}
$$

Add it all up. The result is supposed to be $f(x)$

This adds up to 0

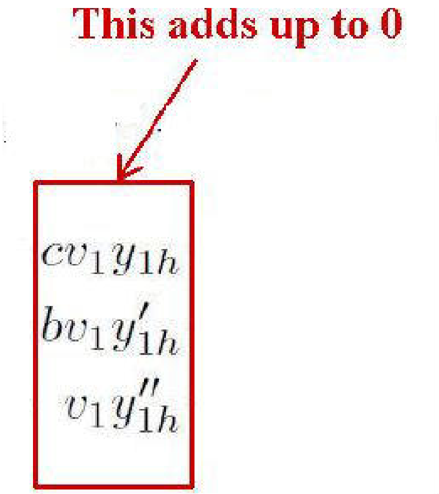
Substitute into $y'' + by' + cy = f(x)$

$$\begin{aligned} cy &= \boxed{cv_1 y_{1h}} + cv_2 y_{2h} \\ by' &= \boxed{bv_1 y'_{1h}} + bv_2 y'_{2h} \\ y'' &= \boxed{v_1 y''_{1h}} + v_2 y''_{2h} + y'_{1h} v'_1 + y'_{2h} v'_2 \end{aligned}$$

Add it all up. The result is supposed to be $f(x)$

Since y_{1h} is supposes to be a homogeneous solution,

$$y''_{1h} + by'_{1h} + cy_{1h} = 0$$



This adds up to 0

Substitute into $y'' + by' + cy = f(x)$

$$
\begin{aligned}
cy &= cv_1 y_{1h} + cv_2 y_{2h} \\
by' &= bv_1 y'_{1h} + bv_2 y'_{2h} \\
y'' &= v_1 y''_{1h} + v_2 y''_{2h} + y'_{1h} v'_1 + y'_{2h} v'_2
\end{aligned}
$$

Add it all up. The result is supposed to be $f(x)$

This adds up to 0

Substitute into $y'' + by' + cy = f(x)$

$$\begin{aligned} cy &= cv_1 y_{1h} + cv_2 y_{2h} \\ by' &= bv_1 y'_{1h} + bv_2 y'_{2h} \\ y'' &= v_1 y''_{1h} + v_2 y''_{2h} + y'_{1h} v'_1 + y'_{2h} v'_2 \end{aligned}$$

$$y'' + by' + cy = 0 + 0 + y'_{1h} v'_1 + y'_{2h} v'_2$$

$$y'_{1h}v'_1 + y'_{2h}v'_2 = f(x)$$

$$y_{1h}v_1' + y_{2h}v_2' = 0$$

$$y_{1h}'v_1' + y_{2h}'v_2' = f(x)$$

$$v_1' = \frac{\begin{vmatrix} 0 & y_{2h} \\ f(x) & y_{2h}' \end{vmatrix}}{\begin{vmatrix} y_{1h} & y_{2h} \\ y_{1h}' & y_{2h}' \end{vmatrix}} \quad v_2' = \frac{\begin{vmatrix} y_{1h} & 0 \\ y_{1h}' & f(x) \end{vmatrix}}{\begin{vmatrix} y_{1h} & y_{2h} \\ y_{1h}' & y_{2h}' \end{vmatrix}}$$

The Wronskian Determinant

$$\mathcal{W} = \begin{vmatrix} y_{1h} & y_{2h} \\ y'_{1h} & y'_{2h} \end{vmatrix}$$

$$\frac{dv_1}{dx} = \frac{1}{\mathcal{W}} \begin{vmatrix} 0 & y_{2h} \\ f(x) & y'_{2h} \end{vmatrix} \qquad \frac{dv_2}{dx} = \frac{1}{\mathcal{W}} \begin{vmatrix} y_{1h} & 0 \\ y'_{1h} & f(x) \end{vmatrix}$$

Example: Find the general solution of:

$$y'' + 4y = x^2$$

$$y_{1h} = \cos 2x \qquad \text{and} \qquad y_{2h} = \sin 2x$$

Find functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ so that

$$y = v_1 \cos 2x + v_2 \sin 2x$$

We will need the Wronskian determinant

$$\begin{aligned} \mathcal{W} &= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} \\ &= 2\cos^2 2x + 2\sin^2 2x \\ &= 2 \end{aligned}$$

$$\frac{dv_1}{dx} = \frac{1}{2}\begin{vmatrix} 0 & \sin 2x \\ x^2 & 2\cos 2x \end{vmatrix} = -\frac{1}{2}x^2 \sin 2x$$

$$\frac{dv_2}{dx} = \frac{1}{2}\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & x^2 \end{vmatrix} = \frac{1}{2}x^2 \cos 2x$$

$$\frac{dv_1}{dx} = \frac{1}{2}\begin{vmatrix} 0 & \sin 2x \\ x^2 & 2\cos 2x \end{vmatrix} = -\frac{1}{2}x^2 \sin 2x$$

$$\begin{aligned} v_1 &= \int -\frac{1}{2}x^2 \sin 2x \, dx \\ &= \frac{1}{4}x^2 \cos 2x - \frac{1}{8}\cos 2x - \frac{1}{4}x \sin 2x + c_1 \end{aligned}$$

$$\frac{dv_2}{dx} = \frac{1}{2}\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & x^2 \end{vmatrix} = \frac{1}{2}x^2 \cos 2x$$

$$\begin{aligned} v_2 &= \int \frac{1}{2}x^2 \cos 2x \, dx \\ &= \frac{1}{4}x^2 \sin 2x - \frac{1}{8}\sin 2x + \frac{1}{4}x \cos 2x + c_2 \end{aligned}$$

$$
\begin{aligned}
y &= v_1 \cos 2x + v_2 \sin 2x \\
&= \left(\frac{1}{4}x^2 \cos 2x - \frac{1}{8}\cos 2x - \frac{1}{4}x \sin 2x + c_1\right)\cos 2x \\
&\qquad + \left(\frac{1}{4}x^2 \sin 2x - \frac{1}{8}\sin 2x + \frac{1}{4}x \cos 2x + c_2\right)\sin 2x \\
&= c_1 \cos 2x + c_2 \sin 2x + \left(\frac{1}{4}x^2 - \frac{1}{8}\right)\left(\cos^2 2x + \sin^2 2x\right) \\
&= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x^2 - \frac{1}{8}
\end{aligned}
$$

$$(D^2 + 4)y = \sec 2x$$

$$(D - 1)^2 y = \frac{1}{x} e^x$$