Differential Equations

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Today's Topic : Variation of Parameters Examples

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Suppose the homogeneous solution is

$y = c_1 y_{1h} + c_2 y_{2h}$ where c_1 and c_2 are constants

Can we find functions

$$v_1 = v_1(x) \quad \text{and} \quad v_2 = v_2(x)$$

so that the nonhomogeneous solution is

$$y = v_1 y_{1h} + v_2 y_{2h}$$

$$v_1' = \frac{\begin{vmatrix} 0 & y_{2h} \\ f(x) & y_{2h}' \end{vmatrix}}{\begin{vmatrix} y_{1h} & y_{2h} \\ y_{1h}' & y_{2h}' \end{vmatrix}} \qquad v_2' = \frac{\begin{vmatrix} y_{1h} & 0 \\ y_{1h}' & f(x) \end{vmatrix}}{\begin{vmatrix} y_{1h} & y_{2h} \\ y_{1h}' & y_{2h}' \end{vmatrix}}$$

The Wronskian Determinant

$$\mathcal{W} = \begin{vmatrix} y_{1h} & y_{2h} \\ y'_{1h} & y'_{2h} \end{vmatrix}$$

$$\frac{dv_1}{dx} = \frac{1}{\mathcal{W}} \begin{vmatrix} 0 & y_{2h} \\ f(x) & y'_{2h} \end{vmatrix} \qquad \frac{dv_2}{dx} = \frac{1}{\mathcal{W}} \begin{vmatrix} y_{1h} & 0 \\ y'_{1h} & f(x) \end{vmatrix}$$

$$y'' + 4y = \sec 2x$$

$$y = v_1 \cos 2x + v_2 \sin 2x$$

We will need the Wronskian determinant

$$\begin{aligned}\mathcal{W} &= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} \\ &= 2\cos^2 2x + 2\sin^2 2x \\ &= 2\end{aligned}$$

$$\frac{dv_1}{dx} = \frac{1}{2}\begin{vmatrix} 0 & \sin 2x \\ \sec 2x & 2\cos 2x \end{vmatrix} = -\frac{1}{2}\sin 2x \sec 2x$$

$$\frac{dv_2}{dx} = \frac{1}{2}\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sec 2x \end{vmatrix} = \frac{1}{2}\sec 2x \cos 2x = \frac{1}{2}$$

$$\frac{dv_1}{dt} = -\frac{1}{2}\sin 2x \sec 2x$$

$$\begin{aligned} v_1 &= \int -\frac{1}{2}\sin 2x \sec 2x \, dx \\ &= -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x}\, dx \\ &= \frac{1}{4}\ln|\cos 2x| + c_1 \end{aligned}$$

$$\frac{dv_2}{dt} = \frac{1}{2}$$

$$v_2 = \int \frac{1}{2}\, dx = \frac{1}{2}x + c_2$$

So our general solution is:

$$y = v_1 \cos 2x + v_2 \sin 2x$$

$$= \left(\frac{1}{4}\ln|\cos 2x| + c_1\right)\cos 2x + \left(\frac{1}{2}x + c_2\right)\sin 2x$$

$$= \underbrace{c_1 \cos 2x + c_2 \sin 2x}_{\text{homogeneous solution}} + \underbrace{\frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x \ln|\cos 2x|}_{\text{particular solution}}$$

Example:

Find the general solution of:

$$(D-1)^2 y = \frac{1}{x} e^x$$

First solve the homogeneous equation:

$$(D-1)^2 y = 0$$

Substitute $y = e^{rx}$

$$(r-1)^2 = 0$$

$$r = 1$$

Solutions are $y_{1h} = e^x$ and $y_{2h} = xe^x$

$$(D-1)^2 y = \frac{1}{x} e^x$$

Find functions v_1 and v_2 so that:

$$y = v_1 e^x + v_2 x e^x$$

$$\mathcal{W} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2z} = e^{2x}$$

$$\frac{dv_1}{dx} = e^{-2x} \begin{vmatrix} 0 & xe^x \\ \frac{1}{x}e^x & (x+1)e^x \end{vmatrix} = -1$$

$$\frac{dv_2}{dx} = e^{-2x} \begin{vmatrix} e^x & 0 \\ e^x & \frac{1}{x}e^x \end{vmatrix} = \frac{1}{x}$$

$$v_1 = \int -1 \, dx = -x + c_1$$

$$v_2 = \int \frac{1}{x} \, dx = \ln|x| + c_2$$

$$y = v_1 e^x + v_2 x e^x = (-x + c_1)e^x + (\ln|x| + c_2)xe^x$$

$$y = v_1 e^x + v_2 x e^x = (-x + c_1)e^x + (\ln|x| + c_2)xe^x$$

Simplify:

$$y = c_1 e^x + (c_2 - 1)xe^x + xe^x \ln|x|$$

If we let $a = c_1$ and $b = c_2 - 1$ we get:

$$y = ae^x + bxe^x + xe^x \ln|x|$$

If $y_{1h},\ y_{2h},\ y_{3h},\ \ldots\ y_{nh}$ are n linearly independent solutions of:

$$\tfrac{d^n y}{dx^n} + a_{n-1}\tfrac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1\tfrac{dy}{dx} + a_0 y = 0$$

then there are functions $v_1,\ v_2,\ \ldots,\ v_n$ so that:

$$y = v_1 y_{1h} + v_2 y_{2h} + \cdots + v_n y_{nh} = \sum_{k=1}^{n} v_k y_{kh}$$

solves the nonhomogeneous equation:

$$\tfrac{d^n y}{dx^n} + a_{n-1}\tfrac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1\tfrac{dy}{dx} + a_0 y = f(x)$$

$$
\frac{dv_k}{dx} = \frac{\begin{vmatrix} y_{1h} & y_{2h} & \cdots & 0 & \cdots & y_{nh} \\ y'_{1h} & y'_{2h} & \cdots & 0 & \cdots & y'_{nh} \\ \vdots & \vdots & & \vdots & & \vdots \\ y_{1h}^{(n-1)} & y_{2h}^{(n-1)} & \cdots & f(x) & \cdots & y_{nh}^{(n-1)} \end{vmatrix}}{\begin{vmatrix} y_{1h} & y_{2h} & \cdots & y_{kh} & \cdots & y_{nh} \\ y'_{1h} & y'_{2h} & \cdots & y'_{kh} & \cdots & y'_{nh} \\ \vdots & \vdots & & \vdots & & \vdots \\ y_{1h}^{(n-1)} & y_{2h}^{(n-1)} & \cdots & y_{kh}^{(n-1)} & \cdots & y_{nh}^{(n-1)} \end{vmatrix}}
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