Today's Topic : Laplace Transforms
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Warmup Exercise:
Calculate the following improper integral.

$$\int_0^\infty e^{-7t}\,dt$$

$$
\begin{aligned}
\int_0^\infty e^{-7t}\,dt &= \lim_{T\to\infty}\int_0^T e^{-7t}\,dt \\
&= \lim_{T\to\infty}\frac{1}{7}\left(1 - e^{-7T}\right) \\
&= \frac{1}{7}
\end{aligned}
$$

Let s be a constant:

$$\int_0^\infty e^{-7t} e^{-st}\, dt$$

$$\begin{aligned}\int_0^\infty e^{-7t} e^{-st}\, dt &= \lim_{T\to\infty} \int_0^T e^{(-s-7)t}\, dt \\ &= \lim_{T\to\infty} \frac{1}{-s-7}\left(e^{-(s+7)T} - 1\right)\end{aligned}$$

We need $s + 7 > 0$ to have convergence

$$\lim_{T\to\infty} e^{-(s+7)T} = \lim_{T\to\infty} \frac{1}{e^{(s+7)T}} = 0$$

$$
\begin{aligned}
\int_0^\infty e^{-7t} e^{-st}\, dt &= \lim_{T \to \infty} \frac{1}{-s-7} \left(e^{-(s+7)T} - 1 \right) \\
&= \frac{1}{-s-7} (0 - 1) \\
&= \frac{1}{s+7}
\end{aligned}
$$

The Laplace transform of $f(t)$:

$$\mathcal{L}\left(f(t)\right) = \int_0^\infty f(t)e^{-st}\,dt$$

Example:

$$\mathcal{L}\left(e^{-7t}\right) = \int_0^\infty e^{-7t}e^{-st}\,dt = \frac{1}{s+7}$$

$$\mathcal{L}\left(e^{-7t}\right) = \frac{1}{s+7}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+7}\right) = e^{-7t}$$

You will need three skills:

a. Compute a Laplace transform
b. Compute an inverse Laplace transform
c. Use transforms to solve differential equations

$$\mathcal{L}\left(e^{-7t}\right) = \frac{1}{s+7}$$

More generally,

$$\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$$

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A special case of $\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$ is when $\lambda = 0$

$$\mathcal{L}(1) = \mathcal{L}\left(e^{0t}\right) = \frac{1}{s-0} = \frac{1}{s}$$

If a and b are constants then:

$$
\begin{aligned}
&\mathcal{L}\left(af(t)+bg(t)\right)=\int_0^\infty (af(t)+bg(t))e^{-st}\,dt\\
&=a\int_0^\infty f(t)e^{-st}\,dt+b\int_0^\infty g(t)e^{-st}\,dt\\
&=a\mathcal{L}\left(f(t)\right)+b\mathcal{L}\left(g(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\left(1 + e^{2t} + 4e^{3t}\right) &= \mathcal{L}\left(1\right) + \mathcal{L}\left(e^{2t}\right) + 4\mathcal{L}\left(e^{3t}\right) \\
&= \frac{1}{s} + \frac{1}{s-2} + \frac{4}{s-3}
\end{aligned}
$$

We are going to need the Laplace transform of t^n

$$\mathcal{L}\left(t^n\right) = \int_0^\infty t^n e^{-st}\,dt$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

$$\mathcal{L}\left(t^1\right) = \int_0^\infty \boxed{t}\,\boxed{e^{-st}\,dt}$$

u ***dv***

$$
\begin{aligned}
\mathcal{L}\left(t^1\right) &= \int_0^\infty te^{-st}\,dt \\
&= \lim_{T\to\infty} \frac{-T}{se^{sT}} + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\
&= 0 + \frac{1}{s}\mathcal{L}\left(1\right) \\
&= \frac{1}{s^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\left(t^2\right) &= \int_0^\infty t^2 e^{-st}\,dt \\
&= \lim_{T\to\infty} \frac{-T^2}{se^{sT}} + \frac{2}{s}\int_0^\infty te^{-st}\,dt \\
&= 0 + \frac{2}{s}\mathcal{L}\left(t\right) \\
&= \frac{2}{s^3}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{T\to\infty} \frac{T^2}{se^{sT}} &= \lim_{T\to\infty} \frac{2T}{s^2e^{sT}} \\
&= \lim_{T\to\infty} \frac{2}{s^3e^{sT}} \\
&= 0
\end{aligned}
$$

$$\lim_{T \to \infty} \frac{T^2}{e^{sT}} = 0$$

$$\lim_{T \to \infty} \frac{T^3}{e^{sT}} = 0$$

$$\lim_{T \to \infty} \frac{T^4}{e^{sT}} = 0$$

$$\vdots$$

$$\lim_{T \to \infty} \frac{T^n}{e^{sT}} = 0$$

$$\lim_{T\to\infty} \frac{f(T)}{e^{sT}} = 0$$

$$
\begin{aligned}
\mathcal{L}\left(f'(t)\right) &= \int_0^\infty e^{-st} f'(t)\,dt \\
&= \lim_{T\to\infty}\left[\frac{f(t)}{e^{st}}\right]_0^T - \int_0^\infty (-s)e^{-st} f(t)\,dt
\end{aligned}
$$

$$\mathcal{L}\left(f'(t)\right) = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv}$$

$$= \underbrace{\lim_{T\to\infty}\left[\frac{f(t)}{e^{st}}\right]_0^T}_{uv} - \underbrace{\int_0^\infty (-s)e^{-st} f(t)\,dt}_{v\,du}$$

$$
\begin{aligned}
\mathcal{L}\left(f'(t)\right) &= \int_0^\infty e^{-st} f'(t)\, dt \\
&= 0 - f(0) + s\int_0^\infty f(t)e^{-st}\, dt \\
&= s\int_0^\infty f(t)e^{-st}\, dt - f(0)
\end{aligned}
$$

$$
\mathcal{L}\left(f'(t)\right) = s\mathcal{L}\left(f(t)\right) - f(0)
$$

$$\mathcal{L}\left(f'(t)\right) = s\mathcal{L}\left(f(t)\right) - f(0)$$

For example, if $f(t) = t^n$ then $f'(t) = nt^{n-1}$ so:

$$\mathcal{L}\left(nt^{n-1}\right) = s\mathcal{L}\left(t^n\right) - 0^n = s\mathcal{L}\left(t^n\right)$$

$$n\mathcal{L}\left(t^{n-1}\right) = s\mathcal{L}\left(t^n\right)$$

$$\mathcal{L}\left(t^n\right) = \frac{n}{s}\mathcal{L}\left(t^{n-1}\right)$$

$$\mathcal{L}\left(t^{n}\right)=\frac{n}{s}\mathcal{L}\left(t^{n-1}\right)$$

$$\mathcal{L}\left(t^{1}\right)=\frac{1}{s}\mathcal{L}\left(t^{0}\right)=\frac{1}{s}\mathcal{L}\left(1\right)=\frac{1}{s}\cdot\frac{1}{s}=\frac{1}{s^{2}}$$

$$\mathcal{L}\left(t^{2}\right)=\frac{2}{s}\mathcal{L}\left(t^{1}\right)=\frac{2}{s}\cdot\frac{1}{s^{2}}=\frac{2}{s^{3}}$$

$$\mathcal{L}\left(t^{3}\right)=\frac{3}{s}\mathcal{L}\left(t^{2}\right)=\frac{3}{s}\mathcal{L}\left(t^{2}\right)=\frac{(3)(2)}{s^{4}}$$

$$\mathcal{L}\left(t^{4}\right)=\frac{4}{s}\mathcal{L}\left(t^{3}\right)=\frac{4}{s}\cdot\frac{(3)(2)}{s^{4}}=\frac{(4)(3)(2)}{s^{5}}$$

$$\mathcal{L}\left(t^2\right) = \frac{2}{s}\mathcal{L}\left(t^1\right) = \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2!}{s^3}$$

$$\mathcal{L}\left(t^3\right) = \frac{3}{s}\mathcal{L}\left(t^2\right) = \frac{3}{s}\mathcal{L}\left(t^2\right) = \frac{3!}{s^4}$$

$$\mathcal{L}\left(t^4\right) = \frac{4}{s}\mathcal{L}\left(t^3\right) = \frac{4}{s} \cdot \frac{(3)(2)}{s^4} = \frac{4!}{s^5}$$

General Formula: $\mathcal{L}\left(t^n\right) = \dfrac{n!}{s^{n+1}}$

General Formula: $$\mathcal{L}\left(t^{n}\right)=\frac{n!}{s^{n+1}}$$

Example:

Calculate the Laplace transform of $2t^2-3t+4$

$$\begin{aligned}\mathcal{L}\left(2t^{2}-3t+4\right)&=2\mathcal{L}\left(t^{2}\right)-3\mathcal{L}\left(t\right)+4\mathcal{L}\left(1\right)\\&=2\cdot\frac{2}{s^{3}}-3\cdot\frac{1}{s^{2}}+4\cdot\frac{1}{s}\\&=\frac{4}{s^{3}}-\frac{3}{s^{2}}+\frac{4}{s}\end{aligned}$$

$$\mathcal{L}\left(f'\right) = s\mathcal{L}\left(f\right) - f(0)$$

This is an algebraic equation relating $\mathcal{L}\left(f'\right)$ to $\mathcal{L}\left(f\right)$

We can use this to reduce a differential equation to an algebraic equation

Find a function $y = y(t)$ that solves the differential equation:

$$y'' - 2y' = e^{2t} \quad \text{where } y(0) = y'(0) = 0$$

Find a function $Y = Y(s)$ that solves the algebraic equation:

$$(s^2 - 2s)Y = \frac{1}{s - 2}$$