# Today's Topic : More Laplace Transforms

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The Laplace transform of $f(t)$ :

$$\mathcal{L}\left(f(t)\right) = \int_0^\infty f(t)e^{-st}\,dt$$

$$\begin{aligned}
\mathcal{L}\left(f(t)\right) &= \int_0^\infty e^{-st} f(t)\,dt \\
\mathcal{L}\left(f'\right) &= s\mathcal{L}\left(f\right) - f(0) \\
\mathcal{L}\left(1\right) &= \frac{1}{s} \\
\mathcal{L}\left(e^{\lambda t}\right) &= \frac{1}{s-\lambda} \\
\mathcal{L}\left(t^n\right) &= \frac{n!}{s^{n+1}}
\end{aligned}$$

Start with a differential equation:

$$a_2 y'' + a_1 y' + a_0 y = f(t)$$

Take the Laplace transform of both sides:

$$\mathcal{L}\left(a_2 y'' + a_1 y' + a_0 y\right) = \mathcal{L}\left(f(t)\right)$$

$$\downarrow$$

We get an equation that's easier to solve

$$\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$$

$$\mathcal{L}\left(e^{i\omega t}\right) = \frac{1}{s-i\omega}$$

$$\mathcal{L}\left(e^{-i\omega t}\right) = \frac{1}{s+i\omega}$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$e^{-i\omega t} = \cos\omega t - i\sin\omega t$$

Add:

$$e^{i\omega t} + e^{-i\omega t} = 2\cos\omega t$$

$$\mathcal{L}\left(e^{i\omega t}\right) + \mathcal{L}\left(e^{-i\omega t}\right) = \mathcal{L}\left(2\cos\omega t\right)$$

$$\frac{1}{s - i\omega} + \frac{1}{s + i\omega} = 2\mathcal{L}\left(\cos\omega t\right)$$

$$
\begin{aligned}
\mathcal{L}\left(\cos \omega t\right) &= \frac{1}{2}\left(\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right) \\
&= \frac{1}{2}\left(\frac{s + i\omega + s - i\omega}{(s - i\omega)(s + i\omega)}\right) \\
&= \frac{s}{s^2 + \omega^2}
\end{aligned}
$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$e^{-i\omega t} = \cos\omega t - i\sin\omega t$$

Subtract:

$$e^{i\omega t} - e^{-i\omega t} = 2i\sin\omega t$$

$$\mathcal{L}\left(e^{i\omega t}\right) - \mathcal{L}\left(e^{-i\omega t}\right) = \mathcal{L}\left(2i\sin\omega t\right)$$

$$\frac{1}{s - i\omega} - \frac{1}{s + i\omega} = 2i\mathcal{L}\left(\sin\omega t\right)$$

$$\mathcal{L}\left(\sin \omega t\right) = \frac{1}{2i}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$$
$$\mathcal{L}\left(t^{n}\right) = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}\left(\cos\omega t\right) = \frac{s}{s^{2}+\omega^{2}}$$
$$\mathcal{L}\left(\sin\omega t\right) = \frac{\omega}{s^{2}+\omega^{2}}$$

$$\mathcal{L}\left(t^{n}e^{\lambda t}\right) = \int_{0}^{\infty} t^{n}e^{\lambda t}e^{-st}\,dt = \int_{0}^{\infty} t^{n}e^{-(s-\lambda)t}\,dt$$

$$\mathcal{L}\left(t^{n}\right)=\int_{0}^{\infty} t^{n} e^{-st}\, dt=\frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left(t^{n} e^{\lambda t}\right)=\int_{0}^{\infty} t^{n} e^{-(s-\lambda) t}\, dt=\frac{n!}{(s-\lambda)^{n+1}}$$

$$
\begin{aligned}
\mathcal{L}\left(e^{\lambda t}\sin\omega t\right) &= \int_0^\infty e^{\lambda t}\sin\omega t \;\cdot\; e^{-st}\,dt \\
&= \int_0^\infty \sin\omega t \cdot e^{-(s-\lambda)t}\,dt
\end{aligned}
$$

$$\mathcal{L}\left(\sin \omega t\right) = \int_0^{\infty} \sin \omega t \cdot e^{-st}\, dt = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}\left(e^{\lambda t} \sin \omega t\right) = \int_0^{\infty} \sin \omega t \cdot e^{-(s-\lambda)t}\, dt = \frac{\omega}{(s-\lambda)^2 + \omega^2}$$

$$\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$$

$$\mathcal{L}\left(t^{n}\right) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left(t^{n} e^{\lambda t}\right) = \frac{n!}{(s-\lambda)^{n+1}}$$

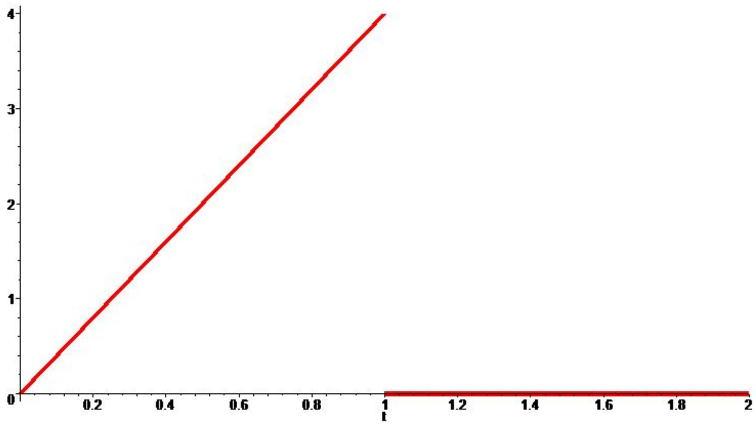
$$\mathcal{L}\left(\sin \omega t\right) = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left(\sin \omega t \cdot e^{\lambda t}\right) = \frac{\omega}{(s-\lambda)^2+\omega^2}$$

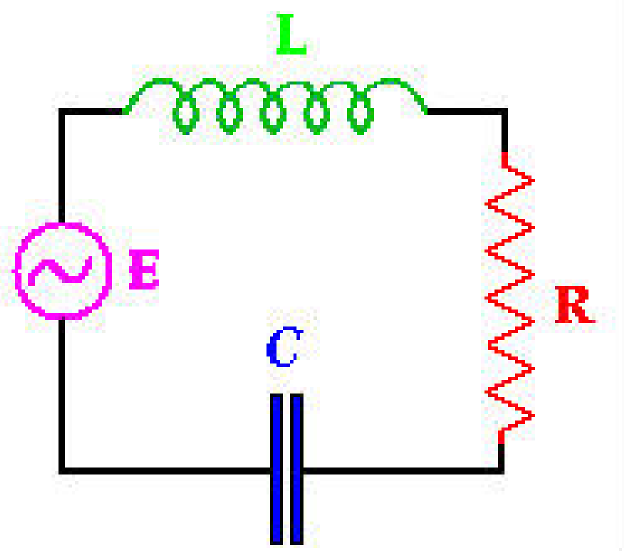
$$\mathcal{L}\left(\cos \omega t\right) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\left(\cos \omega t \cdot e^{\lambda t}\right) = \frac{s-\lambda}{(s-\lambda)^2+\omega^2}$$

$$f(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t > 1 \end{cases}$$



$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}(t)$$



$$f(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t > 1 \end{cases}$$

$$\begin{aligned} \mathcal{L}\left(f(t)\right) &= \int_0^1 f(t)e^{-st}\,dt + \int_1^\infty f(t)e^{-st}\,dt \\ &= \int_0^1 te^{-st}\,dt + \int_1^\infty 0 \cdot e^{-st}\,dt \\ &= \frac{1}{s^2} - \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} \end{aligned}$$