Differential Equations
Today's Topic: Inverse Laplace Transforms
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$$\mathcal{L}\left(e^{\lambda t}t^{n}\right) = \frac{n!}{(s-\lambda)^{n+1}}$$

$$\mathcal{L}\left(\sin \omega t\right) = \frac{\omega}{s^{2}+\omega^{2}}$$

$$\mathcal{L}\left(\cos \omega t\right) = \frac{s}{s^{2}+\omega^{2}}$$

$$\mathcal{L}\left(e^{\lambda t}\sin \omega t\right) = \frac{\omega}{(s-\lambda)^{2}+\omega^{2}}$$

$$\mathcal{L}\left(e^{\lambda t}\cos \omega t\right) = \frac{s-\lambda}{(s-\lambda)^{2}+\omega^{2}}$$

If $\mathcal{L}\left(f(t)\right) = F(s)$ then $f(t) = \mathcal{L}^{-1}\left(F(s)\right)$.

$$\mathcal{L}\left(\sin 4t\right) = \frac{4}{s^2+16} \quad \text{so} \quad \sin 4t = \mathcal{L}^{-1}\left(\frac{4}{s^2+16}\right)$$

$$\mathcal{L}\left(e^t t^3\right) = \frac{6}{(s-1)^4} \quad \text{so} \quad e^t t^3 = \mathcal{L}^{-1}\left(\frac{6}{(s-1)^4}\right)$$

$$\mathcal{L}\left(\left(af(t)+bg(t)\right)\right)=a\mathcal{L}\left(f(t)\right)+b\mathcal{L}\left(g(t)\right)$$

Will inverse Laplace transforms also be linear?

Let $F(s) = \mathcal{L}\left(f(t)\right)$ and $G(s) = \mathcal{L}\left(g(t)\right)$.
$f(t) = \mathcal{L}^{-1}\left(F(s)\right)$ and $g(t) = \mathcal{L}^{-1}\left(G(s)\right)$.

$$\begin{aligned}\mathcal{L}\left((af(t) + bg(t))\right) &= a\mathcal{L}\left(f(t)\right) + b\mathcal{L}\left(g(t)\right) \\ &= aF(s) + bG(s)\end{aligned}$$

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$$\begin{aligned}\mathcal{L}^{-1}\left(aF(s) + bG(s)\right) &= af(t) + bg(t)\\ &= a\mathcal{L}^{-1}\left(F(s)\right) + b\mathcal{L}^{-1}\left(G(s)\right)\end{aligned}$$

Example:
Find the inverse Laplace transform of:

$$\frac{1}{s^2} - \frac{2}{s-1}$$

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{s^2}-\frac{2}{s-1}\right)&=\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)-2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\\&=t-2e^t\end{aligned}$$

Example:

Find the inverse Laplace transform of

$$\frac{1}{(s-1)^4}$$

Start with the general formula:

$$\mathcal{L}\left(t^n e^{\lambda t}\right) = \frac{n!}{(s-\lambda)^{n+1}}$$

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$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{(s-1)^4}\right) &= \mathcal{L}^{-1}\left(\frac{1}{6}\cdot\frac{6}{(s-1)^4}\right) \\
&= \frac{1}{6}\mathcal{L}^{-1}\left(\frac{6}{(s-1)^4}\right) \\
&= \frac{1}{6}e^t t^3
\end{aligned}$$

Example:

Find the inverse Laplace transform of

$$\frac{s+2}{s^2+s}$$

Start by checking the table of Laplace transforms:

$$\begin{aligned}
\mathcal{L}\left(e^{\lambda t}t^n\right) &= \frac{n!}{(s-\lambda)^{n+1}} \\
\mathcal{L}\left(\sin \omega t\right) &= \frac{\omega}{s^2+\omega^2} \\
\mathcal{L}\left(\cos \omega t\right) &= \frac{s}{s^2+\omega^2} \\
\mathcal{L}\left(e^{\lambda t}\sin \omega t\right) &= \frac{\omega}{(s-\lambda)^2+\omega^2} \\
\mathcal{L}\left(e^{\lambda t}\cos \omega t\right) &= \frac{s-\lambda}{(s-\lambda)^2+\omega^2}
\end{aligned}$$

Partial Fractions Decomposition:

$$\frac{s+2}{s^2+s} = \frac{s+2}{s(s+1)} = \frac{a}{s} + \frac{b}{s+1}$$

Solve for a and b

$$\frac{s+2}{s(s+1)} = \frac{2}{s} - \frac{1}{s+1}$$

Use linearity to get the inverse Laplace transform:

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{s+2}{s^2+s}\right) &= \mathcal{L}^{-1}\left(\frac{2}{s} - \frac{1}{s+1}\right) \\
&= 2\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)
\end{aligned}$$

$$\mathcal{L}\left(e^{\lambda t}\right) = \frac{1}{s-\lambda}$$

Therefore:

$$\mathcal{L}(1) = \mathcal{L}\left(e^{0t}\right) = \frac{1}{s}$$

$$\mathcal{L}\left(e^{-t}\right) = \frac{1}{s-(-1)} = \frac{1}{s+1}$$

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s+2}{s^2+s}\right) &= \mathcal{L}^{-1}\left(\frac{2}{s}-\frac{1}{s+1}\right)\\ &= 2\mathcal{L}^{-1}\left(\frac{1}{s}\right)-\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)\\ &= 2-e^{-t}\end{aligned}$$

Example:
Find the inverse Laplace transform of

$$\frac{s^2+3}{(s+1)(s^2+1)}$$

Start with a partial fractions decomposition:

$$\frac{s^2+3}{(s+1)(s^2+1)} = \frac{a}{s+1} + \frac{bs+c}{s^2+1}$$

Solve for a, b and c

$$\frac{s^2+3}{(s+1)(s^2+1)} = \frac{2}{s+1} + \frac{-s+1}{s^2+1}$$
$$= \frac{2}{s+1} + \frac{1}{s^2+1} - \frac{s}{s^2+1}$$

Look up the formulas on the Laplace transform table:

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$$

Use linearity to get $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}\left(\frac{s^2+3}{(s+1)(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{2}{s+1}+\frac{1}{s^2+1}-\frac{s}{s^2+1}\right)$$
$$= 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)+\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)-\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$
$$= 2e^{-t}+\sin t-\cos t$$

How does all of this help us to solve differential equations?

$$\mathcal{L}\left(f'(t)\right) = s\mathcal{L}\left(f(t)\right) - f(0)$$

$$
\begin{aligned}
\mathcal{L}\left(f'(t)\right) &= \int_0^\infty e^{-st} f'(t)\,dt \\
&= 0 - f(0) + s\int_0^\infty f(t)e^{-st}\,dt \\
&= s\int_0^\infty f(t)e^{-st}\,dt - f(0)
\end{aligned}
$$

Let $y = f(t)$

$$\mathcal{L}\left(y'\right) = s\mathcal{L}\left(y\right) - y(0)$$

Example:

Use the identity $\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$ to solve the differential equation:

$$\frac{dy}{dt} + y = 2\sin t \qquad \text{where } y(0) = 1$$

The procedure is to take the Laplace transform of both sides of the equation and solve for $\mathcal{L}(y)$.

$$\mathcal{L}\left(y' + y\right) = \mathcal{L}\left(\left(2\sin t\right)\right)$$

$$\mathcal{L}\left(y'\right) + \mathcal{L}\left(y\right) = 2\mathcal{L}\left(\sin t\right)$$

$$s\mathcal{L}\left(y\right) - y(0) + \mathcal{L}\left(y\right) = 2\left(\frac{1}{s^2+1}\right)$$

Note: Your textbook writes $\mathcal{L}\left(y\right)$ as $Y(s)$ so:

$$sY(s) - y(0) + Y(s) = 2\left(\frac{1}{s^2+1}\right)$$

$$s\mathcal{L}\left(y\right) - y(0) + \mathcal{L}\left(y\right) = 2\left(\frac{1}{s^2+1}\right)$$

$$(s+1)\mathcal{L}\left(y\right) = y(0) + \frac{2}{s^2+1} = 1 + \frac{2}{s^2+1} = \frac{s^2+3}{s^2+1}$$

$$\mathcal{L}\left(y\right) = \frac{s^2+3}{(s+1)(s^2+1)}$$

$$\mathcal{L}\left(y\right) = \frac{s^2+3}{(s+1)(s^2+1)}$$

$$y = \mathcal{L}^{-1}\left(\frac{s^2+3}{(s+1)(s^2+1)}\right) = 2e^{-t} + \sin t - \cos t$$