Differential Equations

Dr. E. Jacobs

Today: Transform Solution of Homogeneous Equations

$$y' = 2y + e^{2t} \qquad \text{where } y(0) = 0$$

First step:
Take the Laplace transform of both sides of the equation and use the identity $\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$.

$$\mathcal{L}\left(y'\right) = \mathcal{L}\left(2y + e^{2t}\right)$$

$$s\mathcal{L}\left(y\right) - y(0) = 2\mathcal{L}\left(y\right) + \mathcal{L}\left(e^{2t}\right)$$

$$s\mathcal{L}\left(y\right) - 0 = 2\mathcal{L}\left(y\right) + \frac{1}{s-2}$$

$$s\mathcal{L}\left(y\right) = 2\mathcal{L}\left(y\right) + \frac{1}{s-2}$$

$$s\mathcal{L}\left(y\right) - 2\mathcal{L}\left(y\right) = \frac{1}{s-2}$$

$$(s-2)\mathcal{L}\left(y\right) = \frac{1}{s-2}$$

$$\mathcal{L}\left(y\right) = \frac{1}{(s-2)^2}$$

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Use the formula: $$\mathcal{L}\left(t^n e^{\lambda t}\right) = \frac{n!}{(s-\lambda)^{n+1}}$$

Conclusion:

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right) = te^{2t}$$

Summary of procedure

$$\text{Diff. Eqtn.} \xrightarrow{\text{Solve}} y = y(t)$$

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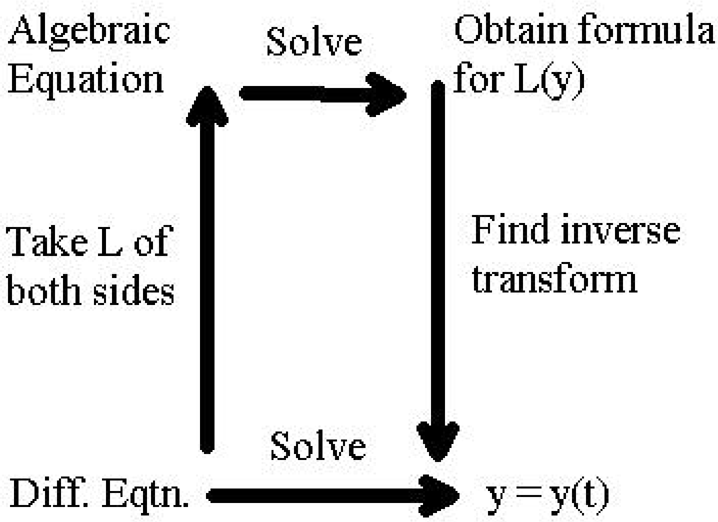


$$\mathcal{L}\left(f'\right) = s\mathcal{L}\left(f\right) - f(0)$$

Replace f by y' and the identity becomes

$$\begin{aligned}
\mathcal{L}\left(y''\right) &= s\mathcal{L}\left(y'\right) - y'(0) \\
&= s(s\mathcal{L}\left(y\right) - y(0)) - y'0) \\
&= s^2\mathcal{L}\left(y\right) - sy(0) - y'(0)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}\left(y'\right) &= s\mathcal{L}\left(y\right) - y(0) \\
\mathcal{L}\left(y''\right) &= s^2\mathcal{L}\left(y\right) - sy(0) - y'(0) \\
\mathcal{L}\left(y'''\right) &= s^3\mathcal{L}\left(y\right) - s^2y(0) - sy'(0) - y''(0)
\end{aligned}
$$

Example: Solve the differential equation.

$$y'' + 2y = 0 \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

We start by taking the Laplace transform of both sides of the equation.

$$\mathcal{L}\left(y''\right) + 2\mathcal{L}\left(y\right) = \mathcal{L}\left(0\right)$$

Use $\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - 0 - 1$

$$\mathcal{L}(y'') + 2\mathcal{L}(y) = \mathcal{L}(0)$$

$$s^2\mathcal{L}(y) - 1 + 2\mathcal{L}(y) = 0$$

$$\mathcal{L}(y) = \frac{1}{s^2+2}$$

$$\mathcal{L}\left(\sin \omega t\right) = \frac{\omega}{s^2 + \omega^2}$$

Now take the inverse Laplace transform:

$$\mathcal{L}\left(y\right) = \frac{1}{s^2 + 2}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{1}{s^2+2}\right) = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left(\frac{\sqrt{2}}{s^2+2}\right) \\ &= \frac{1}{\sqrt{2}}\sin\sqrt{2}t \end{aligned}$$

Example: Solve the differential equation.

$y'' + 2y' + 2y = 0 \qquad$ where $y(0) = 1$ and $y'(0) = -1$

Take $\mathcal{L}$ of both sides and use the identities:
$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - s + 1$
$\mathcal{L}(y') = s\mathcal{L}(y) - y(0) = s\mathcal{L}(y) - 1$

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(0)$$

$$s^2\mathcal{L}(y) - s + 1 + 2(s\mathcal{L}(y) - 1) + 2\mathcal{L}(y) = 0$$

Solve for $\mathcal{L}(y)$

$$s^2\mathcal{L}(y) - s + 1 + 2(s\mathcal{L}(y) - 1) + 2\mathcal{L}(y) = 0$$

$$(s^2 + 2s + 2)\mathcal{L}(y) = s + 1$$

$$\mathcal{L}(y) = \frac{s+1}{s^2 + 2s + 2}$$

$$\mathcal{L}\left(y\right)=\frac{s+1}{s^2+2s+2}$$

Refer to the formula $\mathcal{L}\left(e^{\lambda t}\cos\omega t\right)=\frac{s-\lambda}{(s-\lambda)^2+\omega^2}$

$$y=\mathcal{L}^{-1}\left(\frac{s+1}{s^2+2s+2}\right)=\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)$$

$$=e^{-t}\cos t$$

Example: Solve the following differential equation:

$$y''' - 3y'' + 3y' - y = 0$$

where $y(0) = y'(0) = 0,\ y''(0) = 1$

$$\mathcal{L}(y''') - 3\mathcal{L}(y'') + 3\mathcal{L}(y') - \mathcal{L}(y) = \mathcal{L}(0)$$

Use the identities

$$\begin{aligned}
\mathcal{L}(y') &= s\mathcal{L}(y) - y(0) = s\mathcal{L}(y) \\
\mathcal{L}(y'') &= s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) \\
\mathcal{L}(y''') &= s^3\mathcal{L}(y) - s^2y(0) - sy'(0) - y''(0) \\
&= s^3\mathcal{L}(y) - 1
\end{aligned}$$

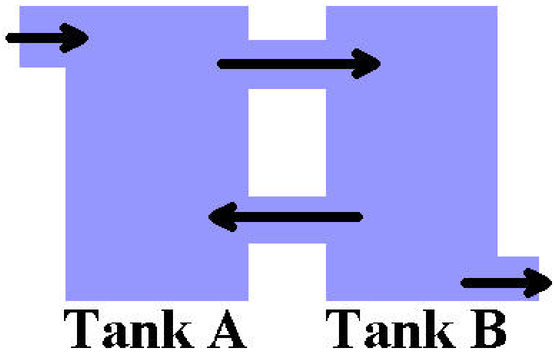
$$s^3\mathcal{L}\left(y\right) - 1 - (s^2\mathcal{L}\left(y\right) + 3s\mathcal{L}\left(y\right) - \mathcal{L}\left(y\right) = 0$$

$$(s^3 - 3s^2 + 3s - 1)\mathcal{L}\left(y\right) = 1$$

$$\mathcal{L}\left(y\right) = \frac{1}{s^3 - 3s^2 + 3s - 1} = \frac{1}{(s-1)^3}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-1)^3}\right) = \frac{1}{2}e^t t^2$$

$$\frac{dx}{dt} = a_{11}x + a_{12}y$$
$$\frac{dy}{dt} = a_{21}x + a_{22}y$$



$$\begin{aligned} x' &= -3x + y \\ y' &= \quad 1x - 3y \end{aligned}$$

Initial conditions: $x(0) = 2$ and $y(0) = 0$

$$\mathcal{L}\left(x'\right) = s\mathcal{L}\left(x\right) - x(0) = s\mathcal{L}\left(x\right) - 2$$

$$\mathcal{L}\left(y'\right) = s\mathcal{L}\left(y\right) - y(0) = s\mathcal{L}\left(y\right) - 0$$

$$
\begin{aligned}
x' &= -3x + y \\
y' &= \quad 1x - 3y
\end{aligned}
$$

Take $\mathcal{L}$ of both sides:

$$
\begin{aligned}
s\mathcal{L}\left(x\right) - 2 &= -3\mathcal{L}\left(x\right) + \mathcal{L}\left(y\right) \\
s\mathcal{L}\left(y\right) - 0 &= \quad \mathcal{L}\left(x\right) - 3\mathcal{L}\left(y\right)
\end{aligned}
$$

$$
\begin{aligned}
s\mathcal{L}(x) - 2 &= -3\mathcal{L}(x) + \mathcal{L}(y) \\
s\mathcal{L}(y) - 0 &= \quad \mathcal{L}(x) - 3\mathcal{L}(y)
\end{aligned}
$$

Regroup terms:

$$
\begin{aligned}
(s+3)\mathcal{L}(x) - \mathcal{L}(y) &= 2 \\
-\mathcal{L}(x) + (s+3)\mathcal{L}(y) &= 0
\end{aligned}
$$

$$
\begin{aligned}
(s+3)\mathcal{L}(x) - \mathcal{L}(y) &= 2 \\
-\mathcal{L}(x) + (s+3)\mathcal{L}(y) &= 0
\end{aligned}
$$

Solve:

$$
\begin{aligned}
\mathcal{L}(x) = \frac{\begin{vmatrix} 2 & -2 \\ 0 & s+3 \end{vmatrix}}{\begin{vmatrix} s+3 & -1 \\ -1 & s+4 \end{vmatrix}} &= \frac{2(s+3)}{s^2+6s+8} \\
&= \frac{2(s+3)}{(s+2)(s+4)}
\end{aligned}
$$

$$
\begin{aligned}
(s+3)\mathcal{L}(x) - \mathcal{L}(y) &= 2 \\
-\mathcal{L}(x) + (s+3)\mathcal{L}(y) &= 0
\end{aligned}
$$

Solve:

$$
\begin{aligned}
\mathcal{L}(y) = \frac{\begin{vmatrix} s+3 & 2 \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} s+3 & -1 \\ -1 & s+4 \end{vmatrix}} &= \frac{2}{s^2+6s+8} \\
&= \frac{2}{(s+2)(s+4)}
\end{aligned}
$$

$$\mathcal{L}\left(x\right) = \frac{2(s+3)}{(s+2)(s+4)} = \frac{1}{s+2} + \frac{1}{s+4}$$

$$\mathcal{L}\left(y\right) = \frac{2}{(s+2)(s+4)} = \frac{1}{s+2} - \frac{1}{s+4}$$

Take the inverse Laplace transform:

$$x(t) = e^{-2t} + e^{-4t}$$

$$y(t) = e^{-2t} - e^{-4t}$$

$$x(t) = e^{-2t} + e^{-4t}$$

$$y(t) = e^{-2t} - e^{-4t}$$

$$\begin{aligned}
\vec{\mathbf{X}} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} e^{-2t} + e^{-4t} \\ e^{-2t} - e^{-4t} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}
\end{aligned}$$

Each term is in the form $\vec{\mathbf{u}}e^{rt}$ where r is an eigenvalue and $\vec{\mathbf{u}}$ is the corresponding eigenvector.

$$ay'' + by' + cy = f(t)$$