Differential Equations
Dr. E. Jacobs
Today: Laplace Solution of Nonhomogeneous Equations

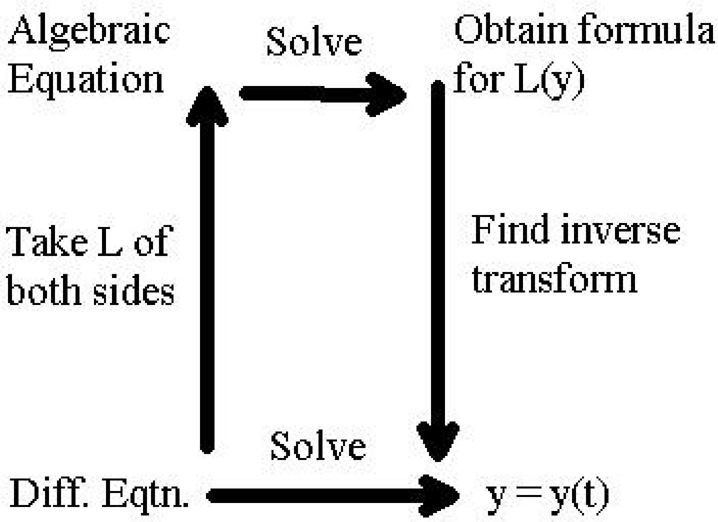
$$\mathcal{L}\left(y'\right) = s\mathcal{L}\left(y\right) - y(0)$$
$$\mathcal{L}\left(y''\right) = s^2\mathcal{L}\left(y\right) - sy(0) - y'(0)$$

Take Laplace transform of both sides of the equation

$$ay'' + by' + cy = f(t)$$

$$a\mathcal{L}\left(y''\right) + b\mathcal{L}\left(y'\right) + c\mathcal{L}\left(y\right) = \mathcal{L}\left(f(t)\right)$$

Summary of procedure



Example:

Solve the following differential equation:

$$(D^2 - 4)y = 3e^t \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

$$\mathcal{L}\left(y'' - 4y\right) = \mathcal{L}\left(3e^t\right)$$

$$s^2\mathcal{L}\left(y\right) - sy(0) - y'(0) - 4\mathcal{L}\left(y\right) = \frac{3}{s-1}$$

$$s^2\mathcal{L}\left(y\right) - s \cdot 0 - 1 - 4\mathcal{L}\left(y\right) = \frac{3}{s-1}$$

$$s^2\mathcal{L}(y) - s \cdot 0 - 1 - 4\mathcal{L}(y) = \frac{3}{s-1}$$

$$(s^2 - 4)\mathcal{L}(y) = 1 + \frac{3}{s-1} = \frac{s+2}{s-1}$$

$$
\begin{aligned}
\mathcal{L}(y) &= \frac{s+2}{(s-1)(s^2-4)} \\
&= \frac{s+2}{(s-1)(s-2)(s+2)} \\
&= \frac{1}{(s-1)(s-2)}
\end{aligned}
$$

$$\mathcal{L}\left(y\right) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}$$

$$y = \mathcal{L}^{-1}\left(\frac{1}{s-2} - \frac{1}{s-1}\right) = e^{2t} - e^{t}$$

Example:

Solve the following differential equation:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 2e^t + 2e^{-t} \qquad \text{where } y(0) = y'(0) = 0$$

$$\mathcal{L}\left(y''\right) + \mathcal{L}\left(y'\right) = 2\mathcal{L}\left(e^t\right) + 2\mathcal{L}\left(e^{-t}\right)$$

$$s^2\mathcal{L}\left(y\right) - sy(0) - y'(0) + s\mathcal{L}\left(y\right) - y(0) = \frac{2}{s-1} + \frac{2}{s+1}$$

$$s^2\mathcal{L}\left(y\right) + s\mathcal{L}\left(y\right) = \frac{2}{s-1} + \frac{2}{s+1} = \frac{4s}{(s-1)(s+1)}$$

$$\mathcal{L}\left(y\right) = \frac{4}{(s-1)(s+1)^2}$$

$$\frac{4}{(s-1)(s+1)^2} = \frac{a}{s-1} + \frac{b}{s+1} + \frac{c}{(s+1)^2}$$

Solve for a, b and c

$$\mathcal{L}\left(y\right) = \frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{(s+1)^2}$$

$$\begin{aligned} y &= \mathcal{L}^{-1}\left(\frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{(s+1)^2}\right) \\ &= e^{t} - e^{-t} - 2te^{-t} \end{aligned}$$

Example:

Solve the following differential equation:

$y'' + 2y' + 2y = 4 + 4t$ where $y(0) = 3$ and $y'(0) = 2$

$$s^2\mathcal{L}(y) - 3s - 2 + 2(s\mathcal{L}(y) - 3) + 2\mathcal{L}(y) = \frac{4}{s} + \frac{4}{s^2}$$

$$s^2\mathcal{L}\left(y\right) - 3s - 2 + 2(s\mathcal{L}\left(y\right) - 3) + 2\mathcal{L}\left(y\right) = \frac{4}{s} + \frac{4}{s^2}$$

$$\mathcal{L}\left(y\right) = \frac{3s^3 + 8s^2 + 4s + 4}{s^2(s^2 + 2s + 2)}$$

This has a partial fractions decomposition

$$s^2\mathcal{L}\left(y\right) - 3s - 2 + 2(s\mathcal{L}\left(y\right) - 3) + 2\mathcal{L}\left(y\right) = \frac{4}{s} + \frac{4}{s^2}$$

$$\begin{aligned}\mathcal{L}\left(y\right) &= \frac{3s^3 + 8s^2 + 4s + 4}{s^2(s^2 + 2s + 2)} \\ &= \frac{2}{s^2} + \frac{6 + 3s}{s^2 + 2s + 2}\end{aligned}$$

$$\mathcal{L}\left(y\right) = \frac{2}{s^2} + \frac{6+3s}{(s+1)^2+1}$$

Compare to:

$$\mathcal{L}\left(e^{\lambda t}\sin \omega t\right) = \frac{\omega}{(s-\lambda)^2+\omega^2}$$

$$\mathcal{L}\left(e^{\lambda t}\cos \omega t\right) = \frac{s-\lambda}{(s-\lambda)^2+\omega^2}$$

$$\mathcal{L}\left(y\right) = \frac{2}{s^2} + \frac{6+3s}{(s+1)^2+1}$$

Compare to:

$$\mathcal{L}\left(e^{-t}\sin t\right) = \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}\left(e^{-t}\cos t\right) = \frac{s+1}{(s+1)^2+1}$$

$$\mathcal{L}(y) = \frac{2}{s^2} + \frac{6 + 3(s + 1 - 1)}{(s+1)^2 + 1}$$

Compare to:

$$\mathcal{L}\left(e^{-t}\sin t\right) = \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}\left(e^{-t}\cos t\right) = \frac{s+1}{(s+1)^2+1}$$

$$\mathcal{L}(y) = \frac{2}{s^2} + \frac{6 + 3(s+1) - 3}{(s+1)^2 + 1}$$

Compare to:

$$\mathcal{L}\left(e^{-t} \sin t\right) = \frac{1}{(s+1)^2 + 1}$$

$$\mathcal{L}\left(e^{-t} \cos t\right) = \frac{s+1}{(s+1)^2 + 1}$$

$$\begin{aligned}\mathcal{L}(y) &= \frac{2}{s^2} + \frac{3 + 3(s+1)}{(s+1)^2 + 1} \\ &= \frac{2}{s^2} + \frac{3}{(s+1)^2 + 1} + \frac{3(s+1)}{(s+1)^2 + 1}\end{aligned}$$

Now take the inverse Laplace transform:

$$y(t) = 2t + 3e^{-t} \sin t + 3e^{-t} \cos t$$