Laplace Transform
of the Unit Step Function

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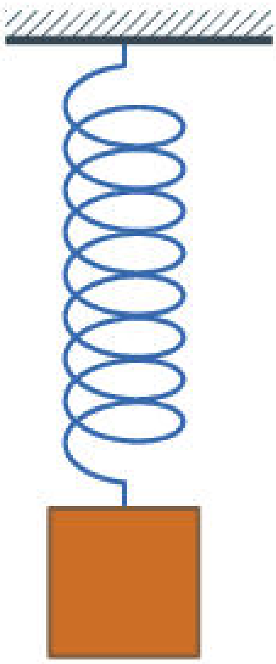
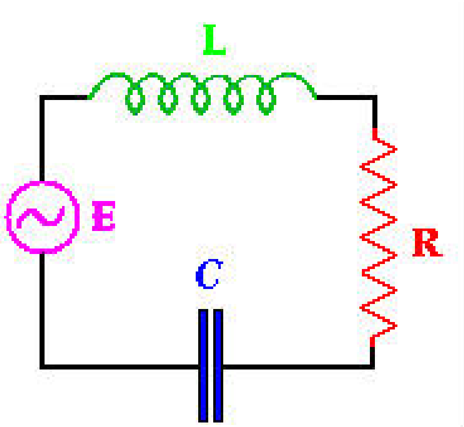
$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$$

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = f(t)$$

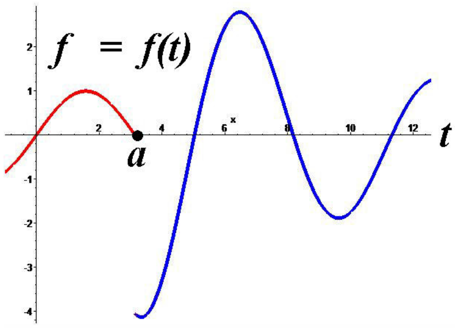


$$L\frac{d^2y}{dt^2} + R\frac{dy}{dt} + \frac{1}{C}y = \mathcal{E}(t)$$



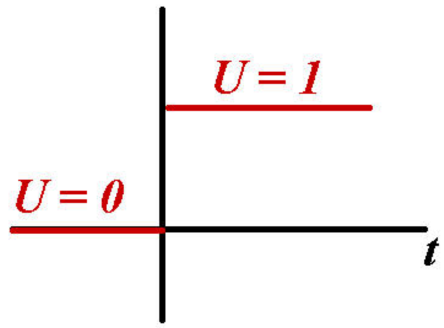
$$\text{Solve} \quad a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = f(t)$$

when $f(t)$ is a discontinuous function



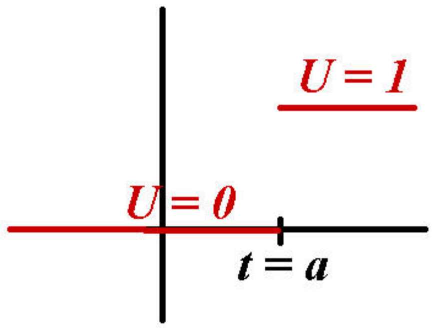
The Unit Step Function

$$\mathcal{U}(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

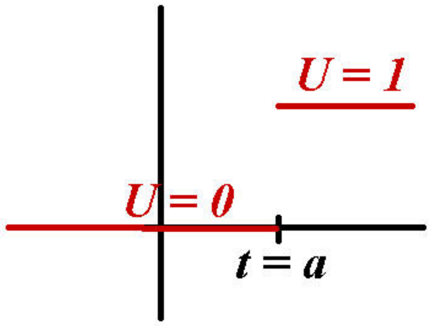


The Shifted Unit Step Function

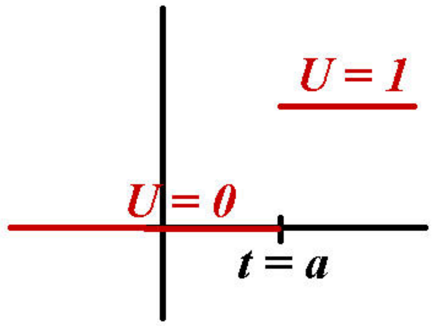
$$\mathcal{U}(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$$



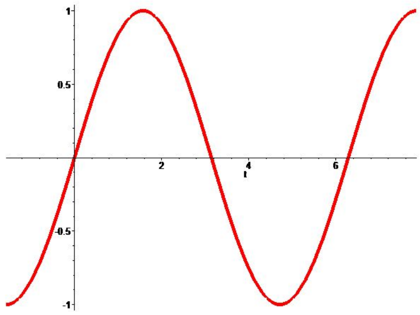
$$\mathcal{L}\left(\mathcal{U}(t-a)\right) = \int_0^\infty \mathcal{U}(t-a)e^{-st}\,dt$$
$$= \int_0^a 0\cdot e^{-st}\,dt + \int_a^\infty 1\cdot e^{-st}\,dt$$



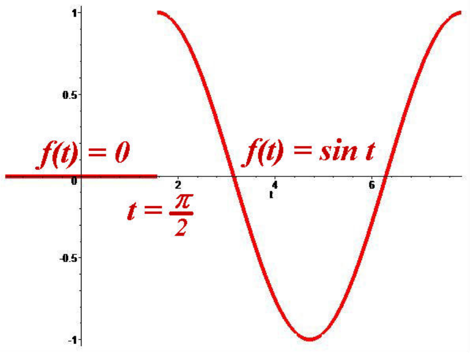
$$\begin{aligned}\mathcal{L}\left(\mathcal{U}(t-a)\right) &= \int_0^\infty \mathcal{U}(t-a)e^{-st}\,dt \\ &= \int_a^\infty 1\cdot e^{-st}\,dt \\ &= e^{-as}\cdot\frac{1}{s}\end{aligned}$$



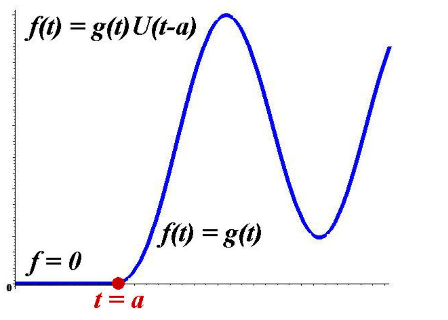
$$y(t) = \sin t$$



$$f(t) = \sin t \cdot \mathcal{U}\left(t - \frac{\pi}{2}\right)$$

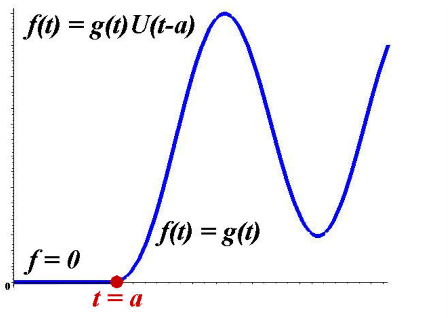


$$f(t) = g(t)\mathcal{U}(t-a)$$



$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) = \int_0^\infty g(t)\mathcal{U}(t-a)e^{-st}\,dt$$

$$= \int_0^a g(t)\cdot 0e^{-st}\,dt + \int_a^\infty g(t)\cdot 1e^{-st}\,dt$$



$$\begin{aligned}\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) &= \int_0^\infty g(t)\mathcal{U}(t-a)e^{-st}\,dt \\ &= \int_a^\infty g(t)e^{-st}\,dt\end{aligned}$$

Let $x = t - a$ so $t = x + a$ and $dt = dx$

$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) = \int_0^\infty g(t)\mathcal{U}(t-a)e^{-st}\,dt$$
$$= \int_a^\infty g(t)e^{-st}\,dt$$

Let $x = t - a$ so $t = x + a$ and $dt = dx$

$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) = \int_0^\infty g(x+a)e^{-s(x+a)}\,dx$$

$$\int_a^b x^3\,dx = \frac{1}{4}b^4 - \frac{1}{4}a^4$$

$$\int_a^b u^3\,du = \frac{1}{4}b^4 - \frac{1}{4}a^4$$

$$\int_a^b t^3\,dt = \frac{1}{4}b^4 - \frac{1}{4}a^4$$

$$
\begin{aligned}
\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) &= \int_0^\infty g(x+a)e^{-s(x+a)}\,dx \\
&= \int_0^\infty g(t+a)e^{-s(t+a)}\,dt \\
&= e^{-sa}\int_0^\infty g(t+a)e^{-st}\,dt \\
&= e^{-sa}\mathcal{L}\left(g(t+a)\right)
\end{aligned}
$$

$$\int_0^\infty g(t)\mathcal{U}(t-a)e^{-st}\,dt = e^{-sa}\int_0^\infty g(t+a)e^{-st}\,dt$$

$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) = e^{-sa}\mathcal{L}\left(g(t+a)\right)$$

$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t+a)\right)$$

$$\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t)\right)$$

Useful for taking Laplace transform of (function)(unit step)

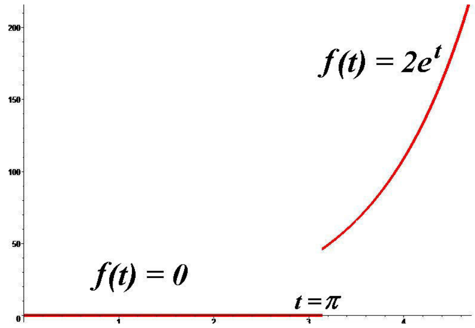
$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t+a)\right)$$

Useful for taking inverse Laplace transforms of $e^{-as}G(s)$ where $G(s) = \mathcal{L}\left(g(t)\right)$

$$\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t)\right)$$

$$f(t) = 2e^t\mathcal{U}(t-\pi)$$

Find the Laplace transform of $f(t)$



$$f(t) = 2e^t\mathcal{U}(t - \pi)$$

Find the Laplace transform of $f(t)$

$$\mathcal{L}\left(g(t)\mathcal{U}(t - a)\right) = e^{-as}\mathcal{L}\left(g(t + a)\right)$$

$$\mathcal{L}\left(2e^t\mathcal{U}(t - \pi)\right) = e^{-\pi s}\mathcal{L}\left(g(t + \pi)\right)$$

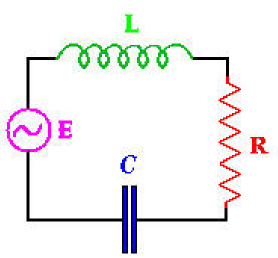
$$f(t) = 2e^t\mathcal{U}(t-\pi)$$

Find the Laplace transform of $f(t)$

$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t+a)\right)$$

$$\begin{aligned}
\mathcal{L}\left(2e^t\mathcal{U}(t-\pi)\right) &= e^{-\pi s}\mathcal{L}\left(g(t+\pi)\right) \\
&= e^{-\pi s}\mathcal{L}\left(2e^{t+\pi}\right) \\
&= 2e^{\pi}e^{-\pi s}\mathcal{L}\left(e^t\right) \\
&= 2e^{\pi}\cdot\frac{e^{-\pi s}}{s-1}
\end{aligned}$$

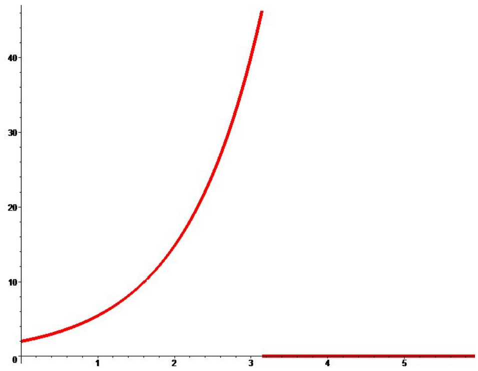
$$L\frac{d^2y}{dt^2} + R\frac{dy}{dt} + \frac{1}{C}y = \mathcal{E}(t)$$



Take $L = 1$, $R = 0$ and $C = 1$

$$y'' + y = \mathcal{E}(t)$$

$$\mathcal{E}(t) = 2e^t(1 - \mathcal{U}(t - \pi))$$



$$y'' + y = 2e^t(1 - \mathcal{U}(t - \pi)) \qquad \text{where } y(0) = y'(0) = 0$$

$$y'' + y = 2e^t - 2e^t\mathcal{U}(t - \pi)$$

$$y'' + y = 2e^t(1 - \mathcal{U}(t - \pi)) \qquad \text{where } y(0) = y'(0) = 0$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}\left(2e^t\right) - \mathcal{L}\left(2e^t\mathcal{U}(t-\pi)\right)$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{2}{s-1} - 2e^{\pi} \cdot \frac{e^{-\pi s}}{s-1}$$

$$s^2\mathcal{L}(y) + \mathcal{L}(y) = \frac{2}{s-1} - 2e^{\pi}e^{-\pi s} \cdot \frac{1}{s-1}$$

$$y'' + y = 2e^t(1 - \mathcal{U}(t - \pi)) \qquad \text{where } y(0) = y'(0) = 0$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}\left(2e^t\right) - \mathcal{L}\left(2e^t\mathcal{U}(t - \pi)\right)$$

$$(s^2 + 1)\mathcal{L}(y) = \frac{2}{s - 1} - 2e^{\pi}e^{-\pi s} \cdot \frac{1}{s - 1}$$

$$\mathcal{L}(y) = \frac{2}{(s - 1)(s^2 + 1)} - e^{\pi}e^{-\pi s} \cdot \frac{2}{(s - 1)(s^2 + 1)}$$

$$\mathcal{L}\left(y\right) = \frac{2}{(s-1)(s^2+1)} - e^{\pi}e^{-\pi s}\cdot\frac{2}{(s-1)(s^2+1)}$$

$$\mathcal{L}\left(y\right) = \frac{1}{s-1} - \frac{(s+1)}{s^2+1} - e^{\pi-\pi s}\left(\frac{1}{s-1} - \frac{(s+1)}{s^2+1}\right)$$

$$
\begin{aligned}
\frac{1}{s-1} - \frac{(s+1)}{s^2+1} &= \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1} \\
&= \mathcal{L}\left(e^t\right) - \mathcal{L}\left(\cos t\right) - \mathcal{L}\left(\sin t\right) \\
&= \mathcal{L}\left(e^t - \cos t - \sin t\right)
\end{aligned}
$$

$$\text{Let } g(t) = e^t - \cos t - \sin t$$

$$
\begin{aligned}
\mathcal{L}\left(y\right) &= \frac{1}{s-1} - \frac{(s+1)}{s^2+1} - e^{\pi - \pi s}\left[\frac{1}{s-1} - \frac{(s+1)}{s^2+1}\right] \\
&= \mathcal{L}\left(g(t)\right) - e^{\pi} e^{-\pi s}\mathcal{L}\left(g(t)\right)
\end{aligned}
$$

where $g(t) = e^t - \cos t - \sin t$

Now use the formula:

$$
\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t)\right)
$$

$$\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t)\right)$$

If $a = \pi$ and $g(t) = e^t - \cos t - \sin t$

$$\begin{aligned} g(t-\pi) &= e^{t-\pi} - \cos(t-\pi) - \sin(t-\pi) \\ &= e^{t-\pi} + \cos t + \sin t \end{aligned}$$

$$\mathcal{L}\left(y\right) = \mathcal{L}\left(g(t)\right) - e^{\pi}e^{-\pi s}\mathcal{L}\left(g(t)\right)$$

where $g(t) = e^{t} - \cos t - \sin t$

Now use the formula:

$$\mathcal{L}\left(g(t-\pi)\mathcal{U}(t-\pi)\right) = e^{-\pi s}\mathcal{L}\left(g(t)\right)$$

$$\mathcal{L}\left(\left(e^{t-\pi} + \cos t + \sin t\right)\mathcal{U}(t-\pi)\right) = e^{-\pi s}\mathcal{L}\left(g(t)\right)$$

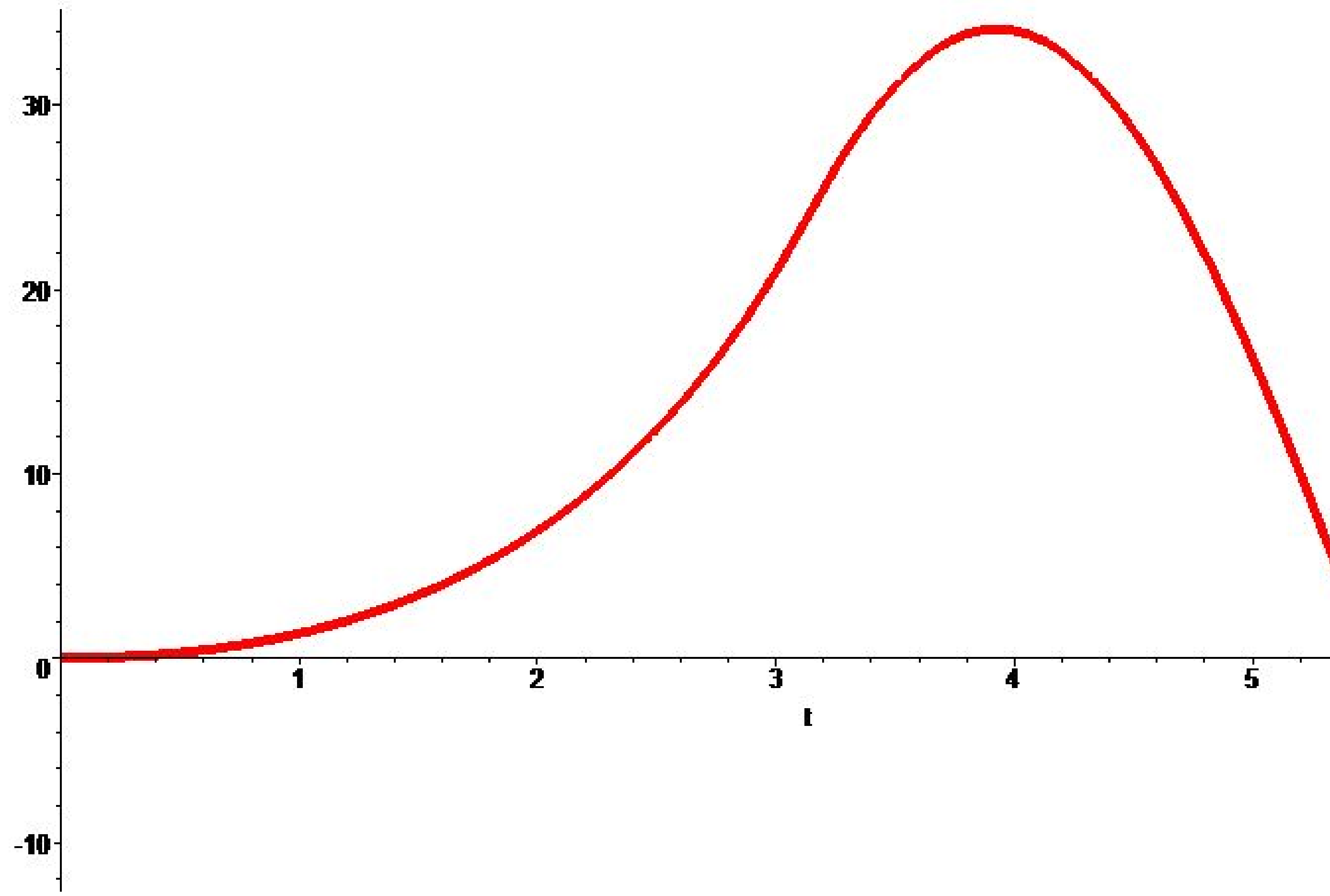
$g(t) = e^t - \cos t - \sin t$

$$\begin{aligned}\mathcal{L}\left(y\right) &= \mathcal{L}\left(g(t)\right) - e^{\pi}\mathcal{L}\left(g(t-\pi)\mathcal{U}(t-\pi)\right)\\ &= \mathcal{L}\left(g(t) - e^{\pi}g(t-\pi)\mathcal{U}(t-\pi)\right)\end{aligned}$$

$$g(t) = e^t - \cos t - \sin t$$

$$\begin{aligned} \mathcal{L}(y) &= \mathcal{L}(g(t)) - e^{\pi}\mathcal{L}(g(t-\pi)\mathcal{U}(t-\pi)) \\ &= \mathcal{L}(g(t) - e^{\pi} g(t-\pi)\mathcal{U}(t-\pi)) \end{aligned}$$

$$\begin{aligned} y &= g(t) - e^{\pi} g(t-\pi)\mathcal{U}(t-\pi) \\ &= e^t - \cos t - \sin t - e^{\pi}\left[e^{t-\pi} + \cos t + \sin t\right]\mathcal{U}(t-\pi) \\ &= e^t - \cos t - \sin t - \left[e^t + e^{\pi}(\cos t + \sin t)\right]\mathcal{U}(t-\pi) \end{aligned}$$



30
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-10
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4
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t