Laplace Transform Problems Involving the Unit Step Function

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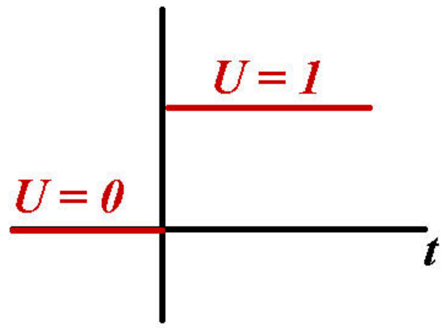
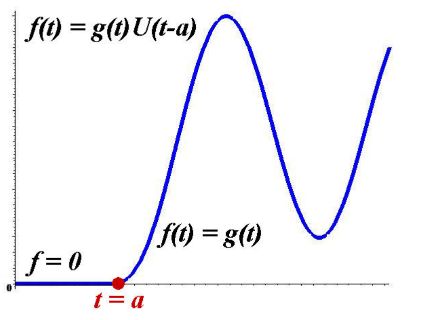
The Unit Step Function

$$\mathcal{U}(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$



$$f(t) = g(t)\mathcal{U}(t-a)$$



$$\mathcal{L}\left(g(t)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t+a)\right)$$

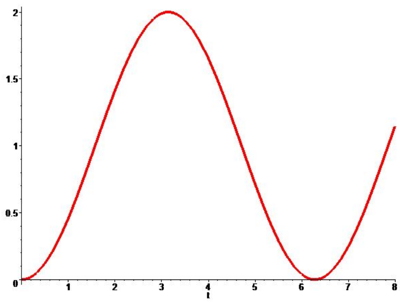
$$\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t)\right)$$

Special case: $g(t) = 1$

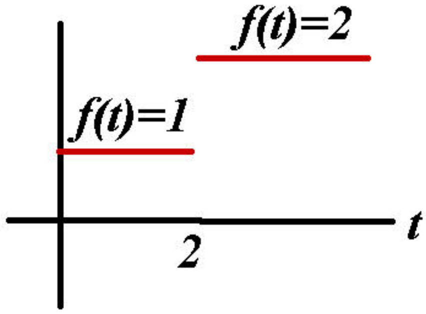
$$\mathcal{L}\left(\mathcal{U}(t-a)\right) = e^{-as} \cdot \frac{1}{s}$$

$$y'' + y = 1 \qquad \text{where } y(0) = y'(0) = 0$$

$$y = 1 - \cos t$$



$$y'' + y = 1 + \mathcal{U}(t-2) \qquad \text{where } y(0) = y'(0) = 0$$



$$y'' + y = 1 + \mathcal{U}(t-2) \qquad \text{where } y(0) = y'(0) = 0$$

$$\mathcal{L}\left(y''\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(1\right) + \mathcal{L}\left(\mathcal{U}(t-2)\right)$$

$$s^2\mathcal{L}\left(y\right) - sy(0) - y'(0) + \mathcal{L}\left(y\right) = \frac{1}{s} + e^{-2s}\frac{1}{s}$$

$$s^2\mathcal{L}\left(y\right) + \mathcal{L}\left(y\right) = \frac{1}{s} + e^{-2s}\frac{1}{s}$$

$$(s^2+1)\mathcal{L}\left(y\right) = \frac{1}{s} + e^{-2s}\frac{1}{s}$$

$$(s^2+1)\mathcal{L}(y) = \frac{1}{s} + e^{-2s}\frac{1}{s}$$

$$\begin{aligned}\mathcal{L}(y) &= \frac{1}{s(s^2+1)} + e^{-2s}\frac{1}{s(s^2+1)} \\ &= \frac{1}{s} - \frac{s}{s^2+1} + e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}(y) &= \frac{1}{s(s^2+1)} + e^{-2s}\frac{1}{s(s^2+1)} \\
&= \frac{1}{s} - \frac{s}{s^2+1} + e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) \\
&= \mathcal{L}(1-\cos t) + e^{-2s}\mathcal{L}(1-\cos t)
\end{aligned}
$$

Use the formula $\mathcal{L}(g(t-a)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t))$ with $a = 2$ and $g(t) = 1 - \cos t$

$$
\begin{aligned}
\mathcal{L}\left(y\right) &= \frac{1}{s(s^2+1)} + e^{-2s}\frac{1}{s(s^2+1)} \\
&= \frac{1}{s} - \frac{s}{s^2+1} + e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) \\
&= \mathcal{L}\left(1-\cos t\right) + e^{-2s}\mathcal{L}\left(1-\cos t\right) \\
&= \mathcal{L}\left(1-\cos t\right) + \mathcal{L}\left((1-\cos(t-2))\mathcal{U}(t-2)\right)
\end{aligned}
$$

$$
y = 1 - \cos t + (1-\cos(t-2))\mathcal{U}(t-2)
$$

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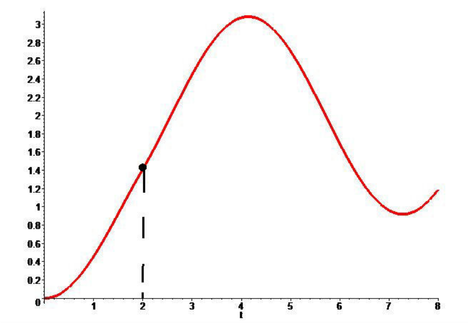
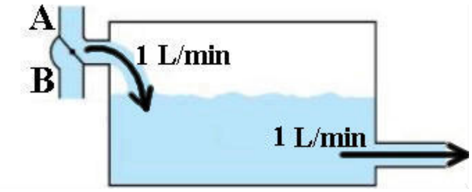
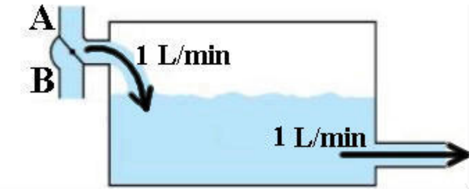


Fluid is flowing into container at 1 liter/min
Fluid is flowing out at 1 liter/min.
For simplicity, assume the container has 1 liter of fluid at all times.

Initially, the tank contains nothing but pure water.



For $t < 2$, fluid coming in has 1 gm of salt/liter.
For $t \geq 2$, fluid coming in has only pure water.



Let $y = y(t)$ be the number of grams of salt in the tank after t minutes.

$$\text{Rate Out} = \frac{1 \text{ liter}}{1 \text{ min}} \cdot \frac{y \text{ grams}}{1 \text{ liter}} = y\ \frac{\text{grams}}{\text{min}}$$

$$\text{Rate In} = \begin{cases} 1 \text{ gram/min} & \text{for } t \leq 2 \\ 0 & \text{for } t > 2 \end{cases} = 1 - \mathcal{U}(t-2)$$

$$\frac{dy}{dt} = \text{Rate in} - \text{Rate out} = 1 - \mathcal{U}(t-2) - y$$

$$y' + y = 1 - \mathcal{U}(t-2)$$

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$$\mathcal{L}(y') + \mathcal{L}(y) = \frac{1}{s} - e^{-2s} \cdot \frac{1}{s}$$

$$s\mathcal{L}\left(y\right) - y(0) + \mathcal{L}y = \frac{1}{s} - e^{-2s} \cdot \frac{1}{s}$$

$$(s+1)\mathcal{L}\left(y\right) = \frac{1}{s} - e^{-2s} \cdot \frac{1}{s}$$

$$\begin{aligned}
\mathcal{L}(y) &= \frac{1}{s(s+1)} - e^{-2s} \cdot \frac{1}{s(s+1)} \\
&= \frac{1}{s} - \frac{1}{s+1} - e^{-2s}\left(\frac{1}{s} - \frac{1}{s+1}\right) \\
&= \mathcal{L}\left(1 - e^{-t}\right) - e^{-2s} \cdot \mathcal{L}\left(1 - e^{-t}\right)
\end{aligned}$$

Use the formula $\mathcal{L}\left(g(t-a)\mathcal{U}(t-a)\right) = e^{-as}\mathcal{L}\left(g(t)\right)$
with $a = 2$ and $g(t) = 1 - e^{-t}$
$g(t-2) = 1 - e^{-(t-2)} = 1 - e^{2-t}$

$$
\begin{aligned}
\mathcal{L}(y) &= \frac{1}{s(s+1)} - e^{-2s} \cdot \frac{1}{s(s+1)} \\
&= \frac{1}{s} - \frac{1}{s+1} - e^{-2s} \left(\frac{1}{s} - \frac{1}{s+1} \right) \\
&= \mathcal{L}\left(1 - e^{-t}\right) - e^{-2s} \cdot \mathcal{L}\left(1 - e^{-t}\right) \\
&= \mathcal{L}\left(1 - e^{-t}\right) - \mathcal{L}\left(\mathcal{U}(t-2)\left(1 - e^{2-t}\right)\right)
\end{aligned}
$$

$$y(t) = 1 - e^{-t} - \mathcal{U}(t-2)\left(1 - e^{2-t}\right)$$