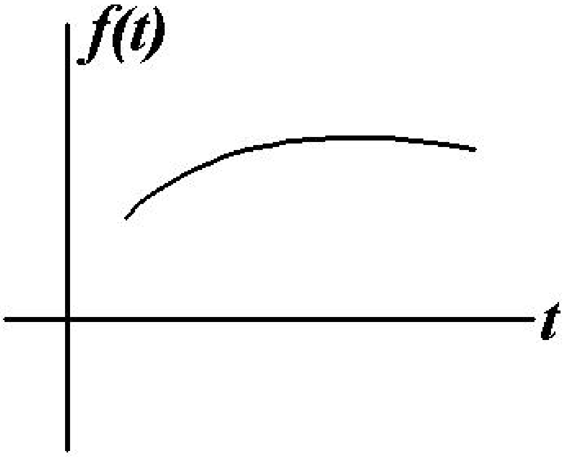
Differential Equations

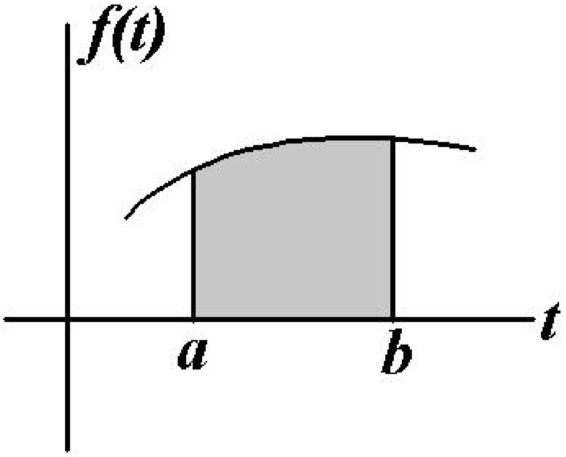
Today's Topic : Unit Impulse

Dr. E. Jacobs

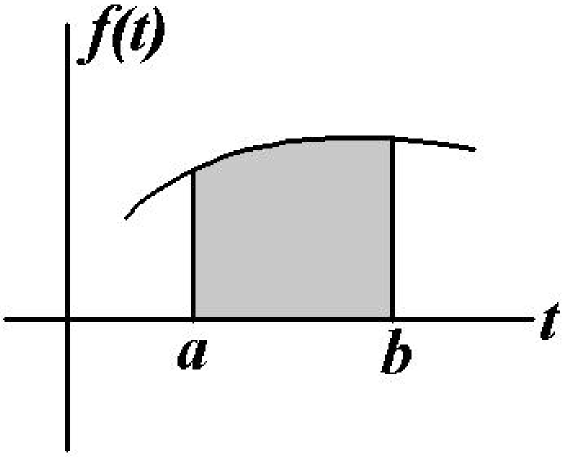
$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = f(t)$$



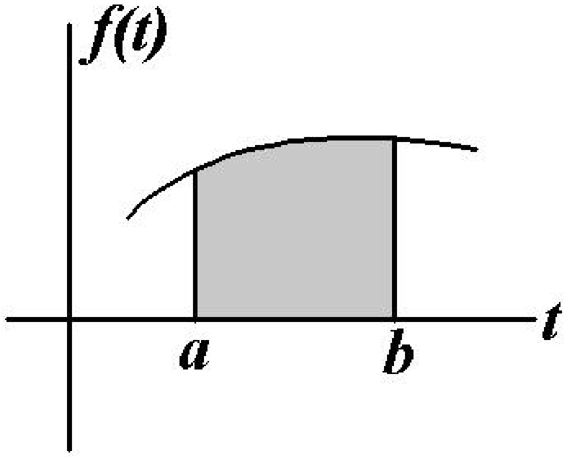
$$\text{Impulse} = \int_a^b f(t)\,dt$$



$$\text{Impulse} = \int_a^b f(t)\,dt = \int_a^b \frac{d}{dt}(mv)\,dt = \Big[mv\Big]_a^b$$



$$\text{Impulse} = \int_a^b f(t)\,dt = \int_a^b \frac{d}{dt}(mv)\,dt = \Delta(mv)$$



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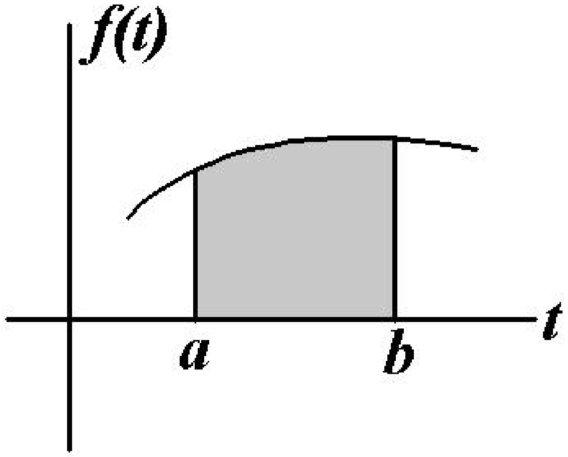
This is a *unit impulse* if $\Delta(mv) = 1$



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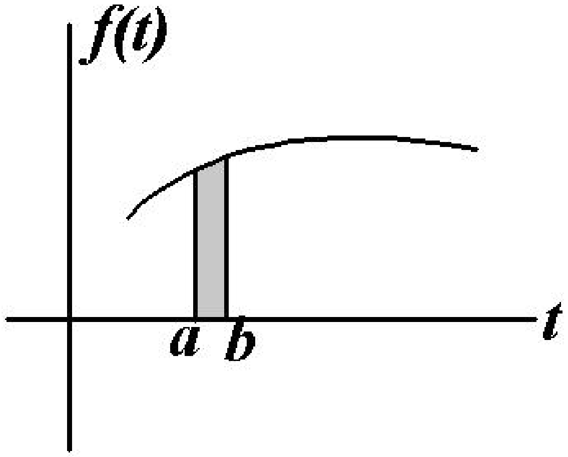

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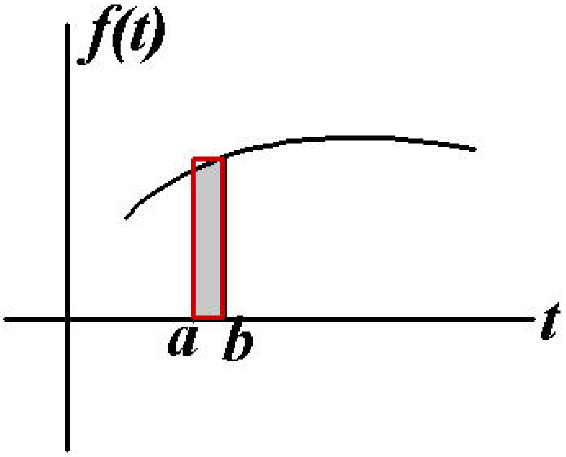
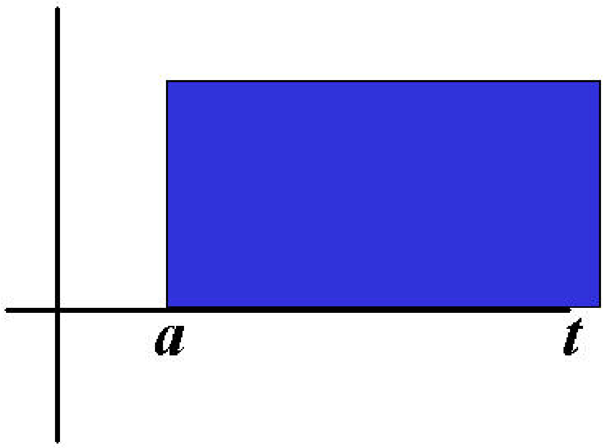
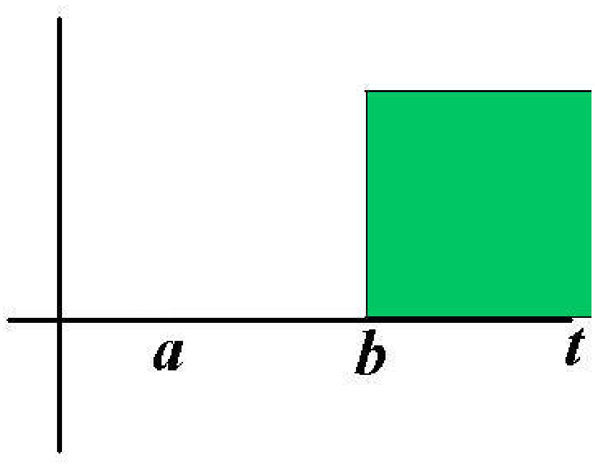


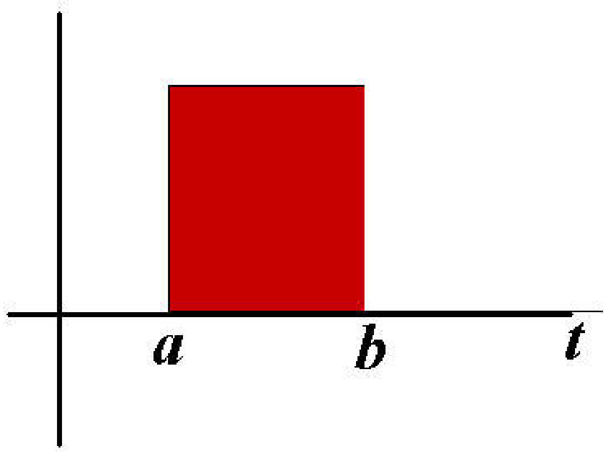
$$\mathcal{U}(t-a)$$



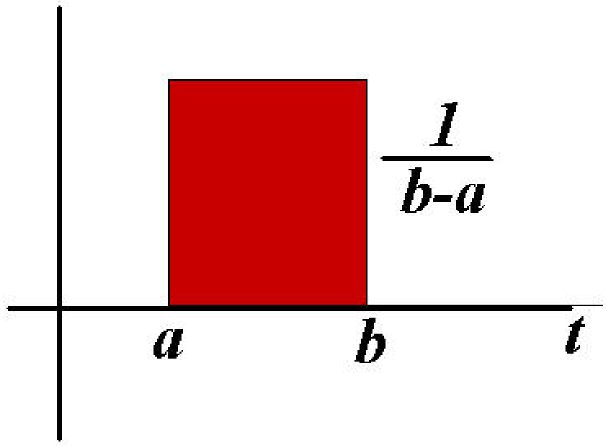
$$\mathcal{U}(t-b)$$



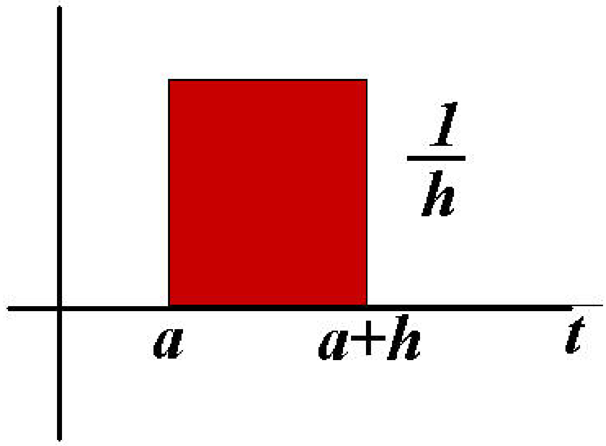
$$\mathcal{U}(t-a)-\mathcal{U}(t-b)$$



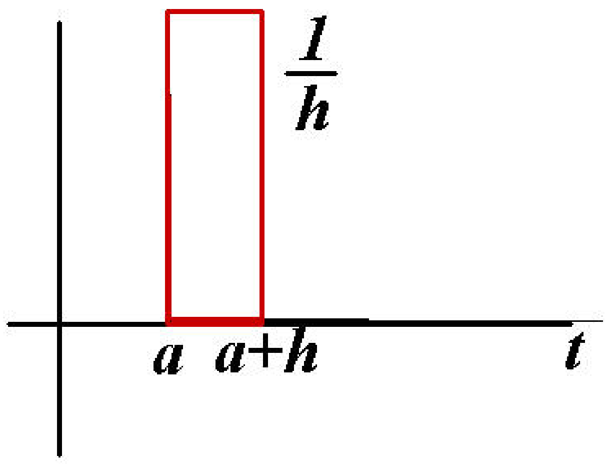
$$\text{Unit Impulse} = \int_a^b \frac{1}{b-a}(\mathcal{U}(t-a) - \mathcal{U}(t-b))\, dt$$



$$\text{Unit Impulse} = \int_a^{a+h} \frac{1}{h}(\mathcal{U}(t-a) - \mathcal{U}(t-a-h))\, dt$$

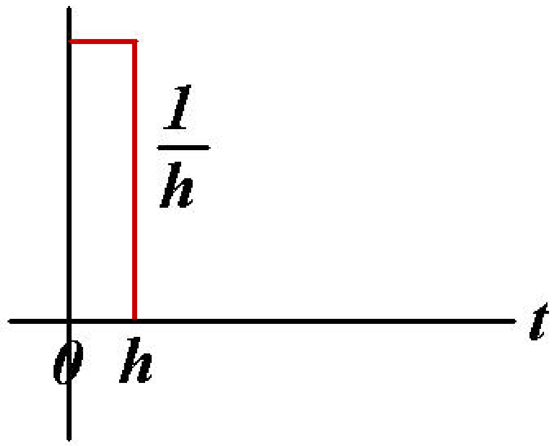


$$\text{Unit Impulse} = \int_a^{a+h} \frac{1}{h}(\mathcal{U}(t-a) - \mathcal{U}(t-a-h))\, dt$$



Solve the following equation:

$$y'' + y = \frac{1}{h}(\mathcal{U}(t) - \mathcal{U}(t-h)) \quad \text{where } y(0) = y'(0) = 0$$



$$y'' + y = \frac{1}{h}(\mathcal{U}(t) - \mathcal{U}(t-h)) \quad \text{where } y(0) = y'(0) = 0$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = \frac{1}{h}(\mathcal{L}(\mathcal{U}(t)) - \mathcal{L}(\mathcal{U}(t-h)))$$

Recall that $\mathcal{L}(\mathcal{U}(t-a)) = e^{-sa}\frac{1}{s}$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{1}{h}\left(\frac{1}{s} - e^{-hs}\frac{1}{s}\right)$$

$$s^2\mathcal{L}\left(y\right) + \mathcal{L}\left(y\right) = \frac{1}{h}\left(\frac{1}{s} - e^{-hs}\frac{1}{s}\right)$$

$$(s^2+1)\mathcal{L}\left(y\right) = \frac{1}{h}\left(\frac{1}{s} - e^{-hs}\frac{1}{s}\right)$$

$$\mathcal{L}\left(y\right) = \frac{1}{h}\left(\frac{1}{s(s^2+1)} - e^{-hs}\frac{1}{s(s^2+1)}\right)$$

$$
\begin{aligned}
\mathcal{L}\left(y\right) &= \frac{1}{h}\left(\frac{1}{s(s^2+1)} - e^{-hs}\frac{1}{s(s^2+1)}\right) \\
&= \frac{1}{h}\left(\frac{1}{s} - \frac{s}{s^2+1} - e^{-hs}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right) \\
&= \frac{1}{h}\big(\mathcal{L}\left(1-\cos t\right) - e^{-hs}\mathcal{L}\left(1-\cos t\right)\big)
\end{aligned}
$$

$$\mathcal{L}(y) = \frac{1}{h}\left(\mathcal{L}(1-\cos t) - e^{-hs}\mathcal{L}(1-\cos t)\right)$$

Use the formula $\mathcal{L}(g(t-a)\mathcal{U}(t-a)) = e^{-sa}\mathcal{L}(g(t))$ where $a = h$ and $g(t) = 1 - \cos t$

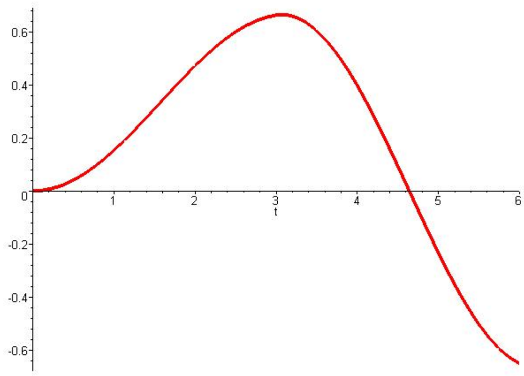
$$\mathcal{L}(y) = \frac{1}{h}(\mathcal{L}(1-\cos t) - \mathcal{L}((1-\cos(t-h))\mathcal{U}(t-h)))$$

$$y(t) = \frac{1}{h}\left(1 - \cos t - (1 - \cos(t - h))\mathcal{U}(t - h)\right)$$

Make h very small

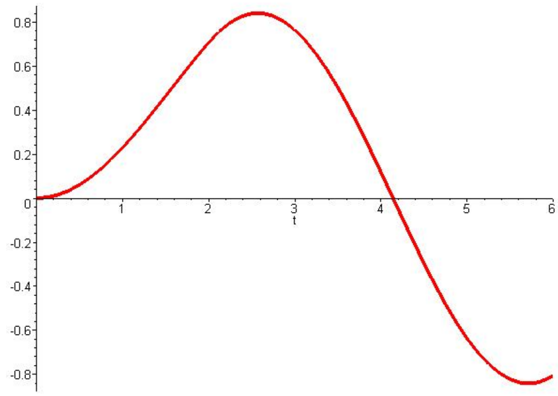
$$y(t) = \frac{1}{h}(1 - \cos t) - \frac{1}{h}(1 - \cos(t-h))\mathcal{U}(t-h)$$

$$h = 3$$



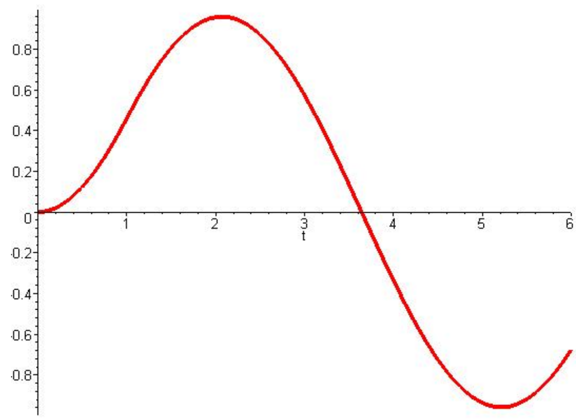
$$y(t) = \frac{1}{h}\,(1 - \cos t) - \frac{1}{h}(1 - \cos(t-h))\mathcal{U}(t-h)$$

$$h = 2$$



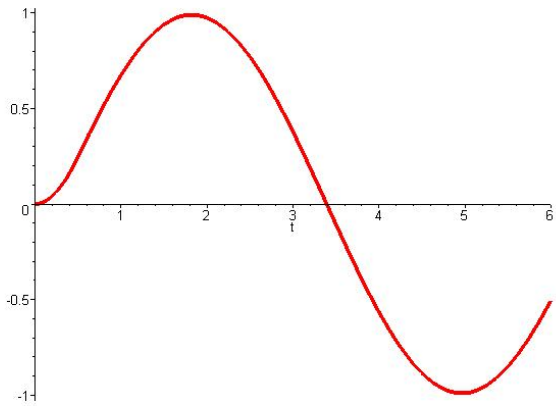
$$y(t) = \frac{1}{h}\,(1 - \cos t) - \frac{1}{h}(1 - \cos(t-h))\mathcal{U}(t-h)$$

$$h = 1$$



$$y(t) = \frac{1}{h}\,(1 - \cos t) - \frac{1}{h}(1 - \cos(t-h))\mathcal{U}(t-h)$$

$$h = 0.5$$



$$y(t) = \frac{1}{h}\,(1 - \cos t) - \frac{1}{h}(1 - \cos(t-h))\mathcal{U}(t-h)$$

For $t > h$, $\mathcal{U}(t-h) = 1$

$$\begin{aligned} y(t) &= \frac{1}{h}(1 - \cos t) - \frac{1}{h}(1 - \cos(t-h)) \\ &= \frac{\cos(t-h) - \cos t}{h} \end{aligned}$$

Use L'hôpital's Rule

$$\lim_{h\to 0}\left(\frac{\cos(t-h)-\cos t}{h}\right) = \lim_{h\to 0}\frac{\sin(t-h)-0}{1}$$
$$= \sin t$$