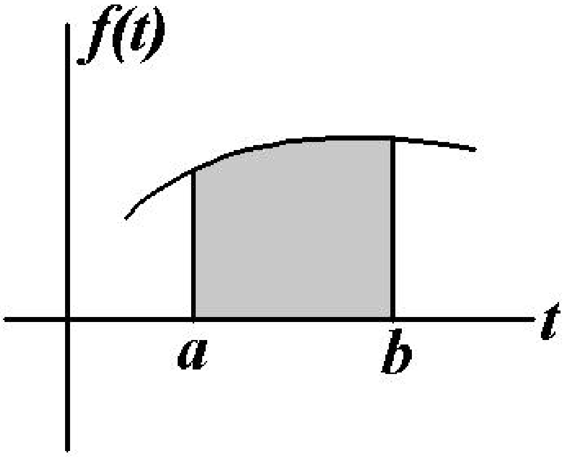
Differential Equations

Today's Topic : The Dirac-Delta Function

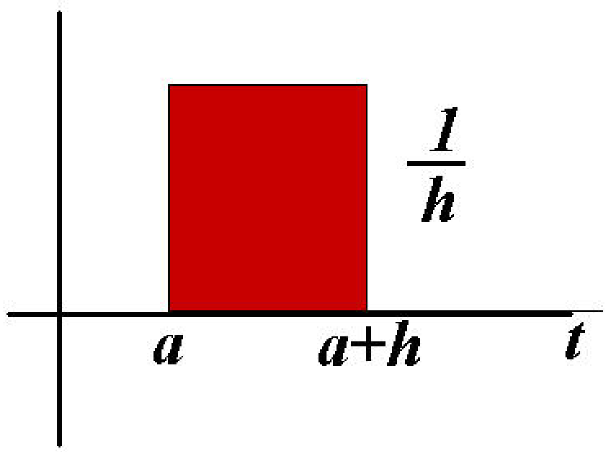
Dr. E. Jacobs

$$\text{Impulse} = \int_a^b f(t)\,dt = \int_a^b \frac{d}{dt}(mv)\,dt = \Delta(mv)$$

If $\Delta(mv) = 1$ then $f(t)$ has produced a unit impulse



$$\text{Unit Impulse} = \int_a^{a+h} \frac{1}{h}(\mathcal{U}(t-a) - \mathcal{U}(t-a-h))\, dt$$



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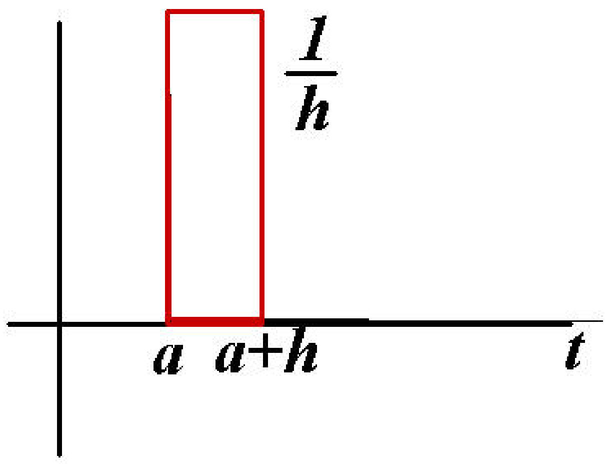
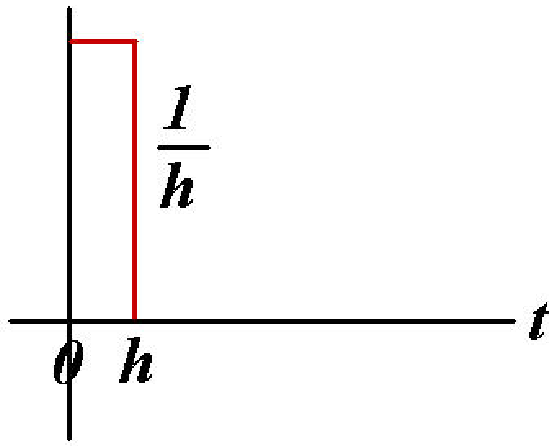


Solve the following equation:

$$y'' + y = \frac{1}{h}(\mathcal{U}(t) - \mathcal{U}(t-h)) \quad \text{where } y(0) = y'(0) = 0$$



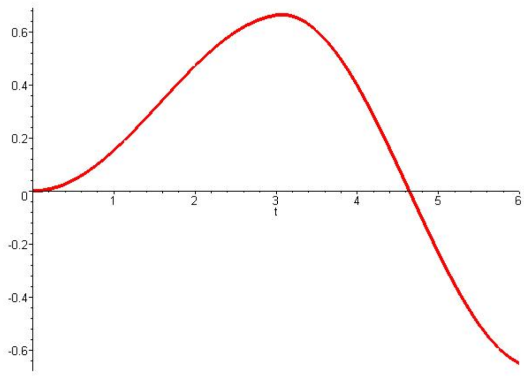
$$y'' + y = \frac{1}{h}(\mathcal{U}(t) - \mathcal{U}(t-h)) \quad \text{where } y(0) = y'(0) = 0$$

$$y(t,h) = \frac{1}{h}\left(1 - \cos t - (1 - \cos(t-h))\mathcal{U}(t-h)\right)$$

Make h very small

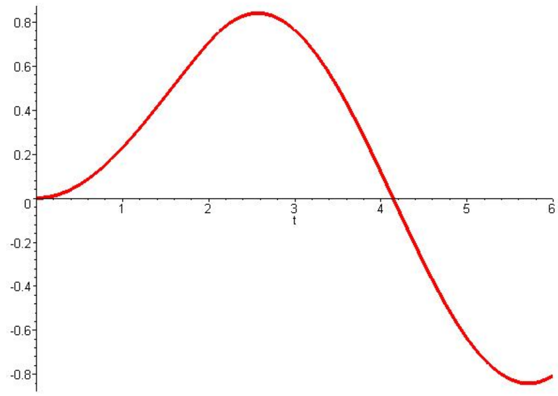
$$y(t,h) = \frac{1}{h}\,(1-\cos t) - \frac{1}{h}(1-\cos(t-h))\mathcal{U}(t-h)$$

$$h = 3$$



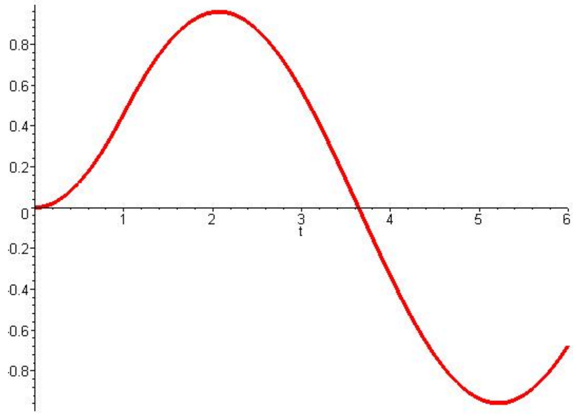
$$y(t.h) = \frac{1}{h}\,(1 - \cos t) - \frac{1}{h}(1 - \cos(t-h))\mathcal{U}(t-h)$$

$$h = 2$$



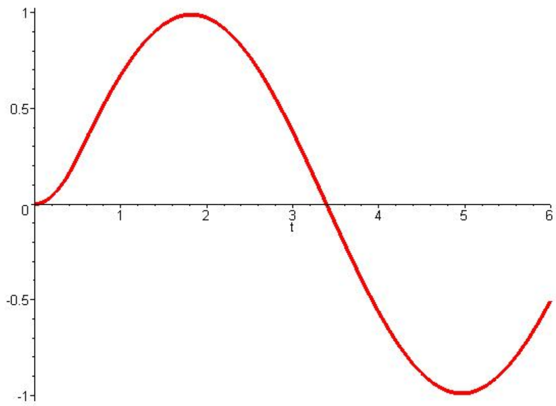
$$y(t,h)=\frac{1}{h}\,(1-\cos t)-\frac{1}{h}(1-\cos(t-h))\mathcal{U}(t-h)$$

$$h=1$$



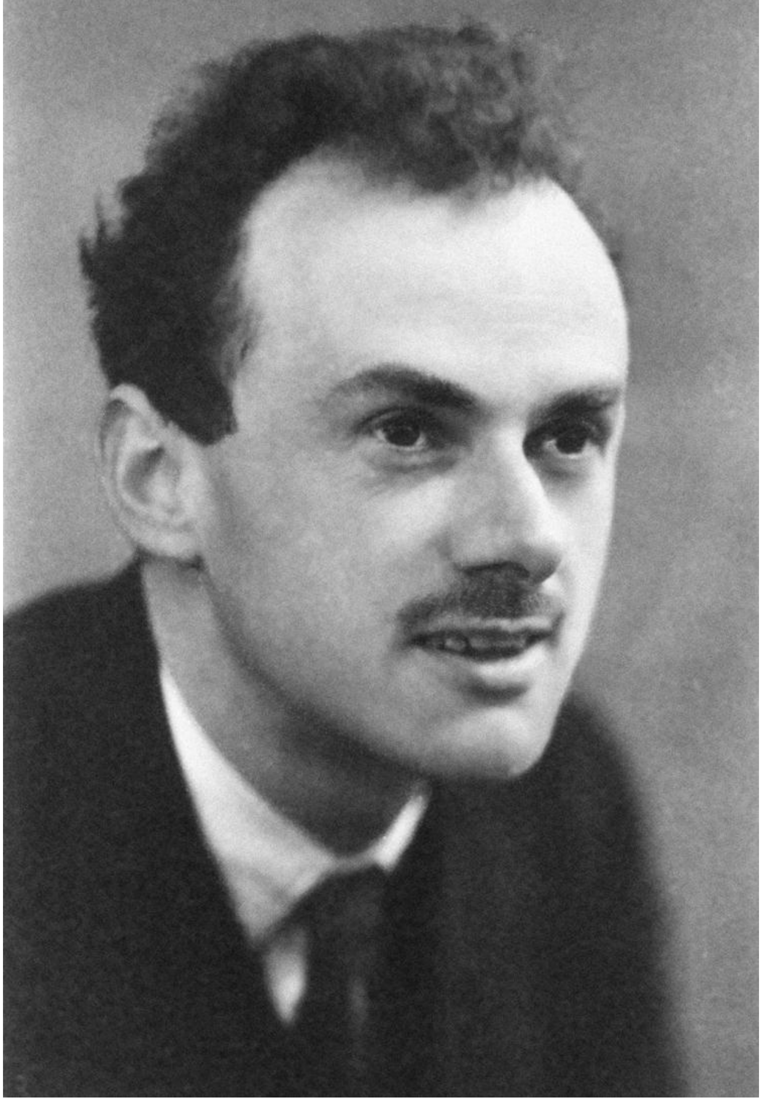
$$y(t,h) = \frac{1}{h}\,(1 - \cos t) - \frac{1}{h}(1 - \cos(t-h))\mathcal{U}(t-h)$$

$$h = 0.5$$



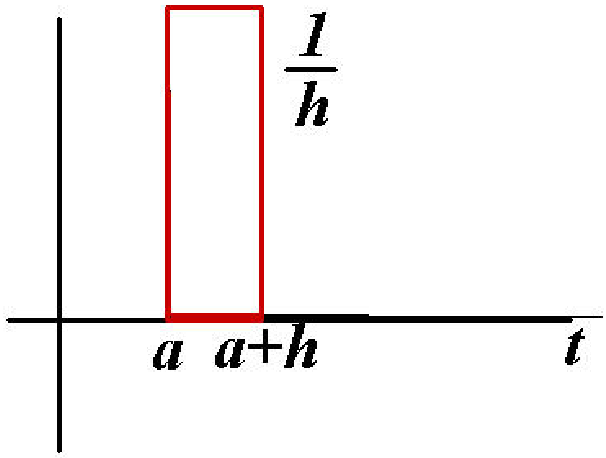
$$y(t,h) = \frac{1}{h}\,(1-\cos t) - \frac{1}{h}(1-\cos(t-h))\mathcal{U}(t-h)$$

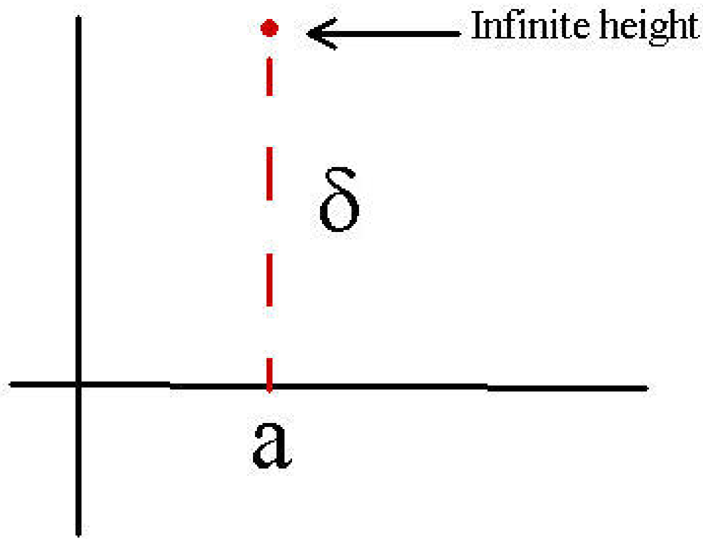
$$\lim_{h\to 0} y(t,h) = \sin t$$



$$\frac{1}{h}(\mathcal{U}(t-a)-\mathcal{U}(t-a-h))$$

Can we take the limit as $h \to 0$?

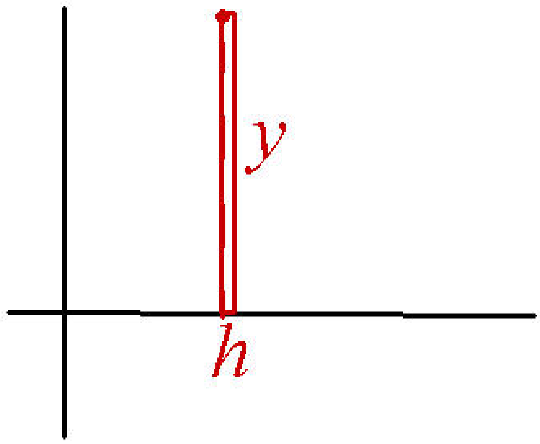




Infinite height
δ
a

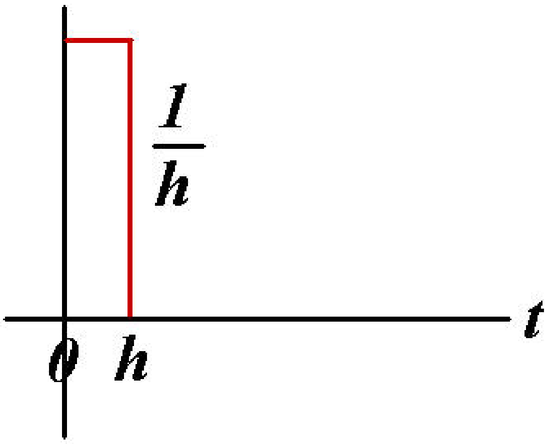
$$\text{Area} = y \cdot h$$

If $h = 0$ then the area is 0

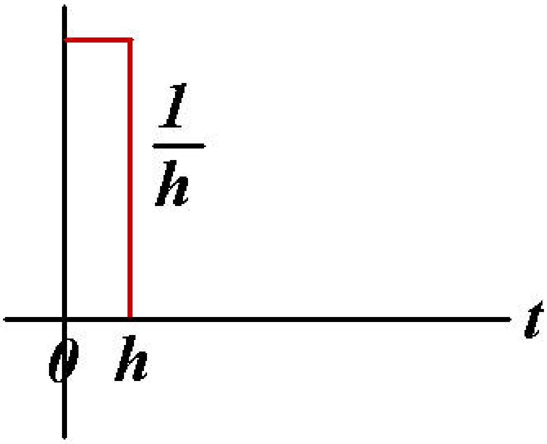


$$\mathcal{L}\left(\frac{1}{h}(\mathcal{U}(t) - \mathcal{U}(t-h))\right)$$

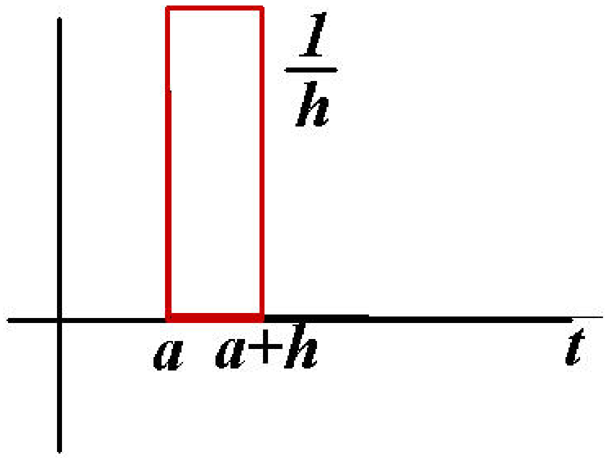
Take the limit as $h \to 0$



$$\mathcal{L}(\delta(t)) = \lim_{h\to 0} \mathcal{L}\left(\frac{1}{h}(\mathcal{U}(t) - \mathcal{U}(t-h))\right)$$



$$\mathcal{L}(\delta(t-a)) = \lim_{h\to 0} \mathcal{L}\left(\frac{1}{h}(\mathcal{U}(t-a) - \mathcal{U}(t-a-h))\right)$$



$$\mathcal{L}\left(\frac{1}{h}(\mathcal{U}(t-a)-\mathcal{U}(t-a-h))\right)$$

$$=\frac{1}{h}\mathcal{L}(\mathcal{U}(t-a))-\frac{1}{h}\mathcal{L}(\mathcal{U}(t-a-h)$$

We know that $\mathcal{L}(\mathcal{U}(t-a))=e^{-as}\frac{1}{s}$
Replace a with $a+h:\ \mathcal{L}(\mathcal{U}(t-a-h))=e^{-as-hs}\frac{1}{s}$

$$
\begin{aligned}
&\mathcal{L}\left(\frac{1}{h}(\mathcal{U}(t-a)-\mathcal{U}(t-a-h))\right)\\
&=\frac{1}{h}\mathcal{L}(\mathcal{U}(t-a))-\frac{1}{h}\mathcal{L}(\mathcal{U}(t-a-h)\\
&=\frac{1}{hs}e^{-as}-\frac{1}{hs}e^{-as-hs}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}\left(\delta(t-a)\right) &= \lim_{h\to 0}\left(\frac{1}{hs}e^{-as} - \frac{1}{hs}e^{-as-hs}\right) \\
&= \frac{1}{s}e^{-as}\lim_{h\to 0}\frac{1-e^{-hs}}{h}
\end{aligned}
$$

Use L'hôpital's Rule

$$\begin{aligned}
\mathcal{L}\left(\delta(t-a)\right) &= \lim_{h\to 0}\left(\frac{1}{hs}e^{-as} - \frac{1}{hs}e^{-as-hs}\right) \\
&= \frac{1}{s}e^{-as}\lim_{h\to 0}\frac{1-e^{-hs}}{h} \\
&= \frac{1}{s}e^{-as}\lim_{h\to 0}\frac{se^{-hs}}{1} \\
&= \frac{1}{s}e^{-as}\cdot\frac{se^{0}}{1} = e^{-as}
\end{aligned}$$

$$\mathcal{L}\left(\delta(t-a)\right) = e^{-as}$$

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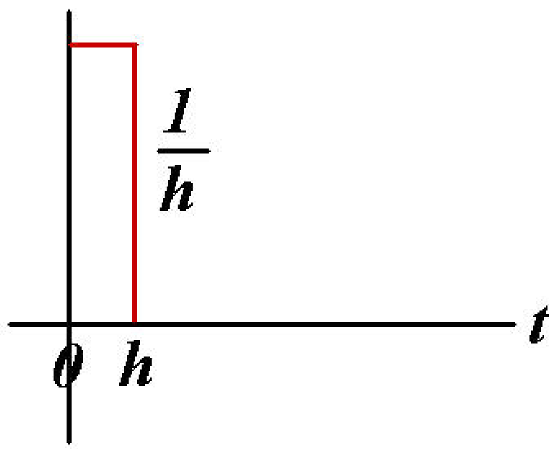
$$\lim_{h\to 0} \mathcal{L}\left(\frac{1}{h}(\mathcal{U}(t-a) - \mathcal{U}(t-a-h))\right) = e^{-as}$$

$$\lim_{h\to 0} \int_0^\infty e^{-st}\frac{1}{h}(\mathcal{U}(t-a) - \mathcal{U}(t-a-h))\,dt = e^{-as}$$

Solve the following equation:

$$y'' + y = \frac{1}{h}(\mathcal{U}(t) - \mathcal{U}(t-h)) \quad \text{where } y(0) = y'(0) = 0$$

Take the limit of the answer as $h \to 0$



Solve the following equation:

$$y'' + y = \frac{1}{h}(\mathcal{U}(t) - \mathcal{U}(t-h)) \quad \text{where } y(0) = y'(0) = 0$$

Take the limit of the answer as $h \to 0$

$$y'' + y = \delta(t)$$

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$$\mathcal{L}\left(y''\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(\delta(t)\right)$$

$$s^2\mathcal{L}\left(y\right) - sy(0) - y'(0) + \mathcal{L}\left(y\right) = \mathcal{L}\left(\delta(t)\right)$$

$$s^2\mathcal{L}\left(y\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(\delta(t)\right)$$

Now use $\mathcal{L}\left(\delta(t-a)\right) = e^{-as}$ with $a = 0$

$$s^2\mathcal{L}\left(y\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(\delta(t)\right)$$

$$(s^2 + 1)\mathcal{L}\left(y\right) = 1$$

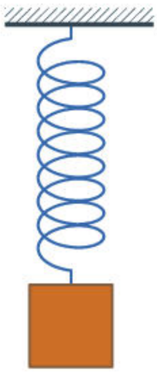
$$\mathcal{L}\left(y\right) = \frac{1}{s^2 + 1}$$

$$y = \sin(t)$$

$$my'' + \beta y' + ky = F(t)$$

$$m = 4 \qquad \beta = 4 \qquad k = 1 \qquad F(t) = \delta(t-6)$$

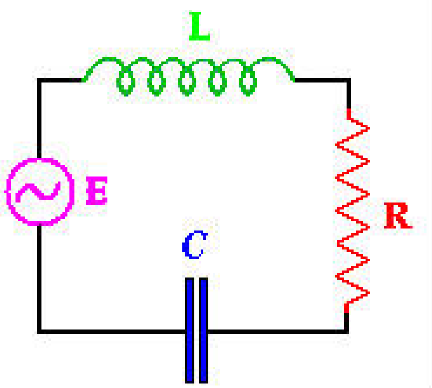
$$4y'' + 4y' + y = \delta(t-6) \qquad y(0) = 0 \qquad y'(0) = 1$$



$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = \mathcal{E}(t)$$

$$L = 4 \qquad R = 4 \qquad C = 1$$

$$4Q'' + 4Q' + Q = \delta(t-6)$$



$$4y'' + 4y' + y = \delta(t-6) \qquad y(0) = 0 \qquad y'(0) = 1$$

$$4\mathcal{L}\left(y''\right) + 4\mathcal{L}\left(y'\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(\delta(t-6)\right)$$

$$\mathcal{L}\left(y''\right) = s^2\mathcal{L}\left(y\right) - sy(0) - y'(0) = s^2\mathcal{L}\left(y\right) - 1$$
$$\mathcal{L}\left(y'\right) = s\mathcal{L}\left(y\right) - y(0) = s\mathcal{L}\left(y\right)$$

$$4s^2\mathcal{L}\left(y\right) - 4 + 4s\mathcal{L}\left(y\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(\delta(t-6)\right)$$

$$(4s^2+4s+1)\mathcal{L}\left(y\right)-4=\mathcal{L}\left(\delta(t-6)\right)$$

Use the formula $\mathcal{L}\left(\delta(t-a)\right)=e^{-as}$ with $a=6$

$$(4s^2+4s+1)\mathcal{L}\left(y\right)-4=e^{-6s}$$

$$\left(s^2+s+\frac{1}{4}\right)\mathcal{L}\left(y\right)=1+\frac{1}{4}e^{-6s}$$

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$$\left(s+\frac{1}{2}\right)^2\mathcal{L}\left(y\right)=1+\frac{1}{4}e^{-6s}$$

$$\mathcal{L}\left(y\right)=\frac{1}{\left(s+\frac{1}{2}\right)^2}+\frac{1}{4}e^{-6s}\frac{1}{\left(s+\frac{1}{2}\right)^2}$$

Use $\mathcal{L}\left(t^n e^{\lambda t}\right)=\frac{n!}{(s-\lambda)^{n+1}}$ with $n=1$ and $\lambda=-\frac{1}{2}$

$$\mathcal{L}(y) = \mathcal{L}\left(te^{-t/2}\right) + \frac{1}{4}e^{-6s}\mathcal{L}\left(te^{-t/2}\right)$$

Use the formula $\mathcal{L}(g(t-a)\mathcal{U}(t-a)) = e^{-as}\mathcal{L}(g(t))$ with $a = 6$ and $g(t) = te^{-t/2}$

$$\mathcal{L}(y) = \mathcal{L}\left(te^{-t/2}\right) + \frac{1}{4}\mathcal{L}\left((t-6)e^{-(t-6)/2}\mathcal{U}(t-6)\right)$$

$$y = te^{-t/2} + \frac{1}{4}(t-6)e^{-(t-6)/2}\mathcal{U}(t-6)$$