Differential Equations

Today's Topic : The Convolution Theorem

Dr. E. Jacobs

$$y'' - y = 0 \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

$$s^2\mathcal{L}\left(y\right) - 1 - \mathcal{L}\left(y\right) = 0$$

$$\mathcal{L}\left(y\right) = \frac{1}{s^2 - 1} = \frac{1}{(s-1)(s+1)}$$

$$\mathcal{L}\left(y\right) = \frac{1}{(s-1)(s+1)} = \frac{\frac{1}{2}}{s-1} - \frac{\frac{1}{2}}{s+1}$$

$$y = \frac{1}{2}\left(e^{t} - e^{-t}\right) = \sinh t$$

$$
\begin{aligned}
\mathcal{L}\left(y\right) &= \frac{1}{(s-1)(s+1)} \\
&= \frac{1}{s-1} \cdot \frac{1}{s+1} \\
&= \mathcal{L}\left(e^{t}\right) \cdot \mathcal{L}\left(e^{-t}\right)
\end{aligned}
$$

$$\mathcal{L}\left(f(t)\right)+\mathcal{L}\left(g(t)\right)=\mathcal{L}\left(f(t)+g(t)\right)$$

What about $\mathcal{L}\left(f(t)\right)\cdot\mathcal{L}\left(g(t)\right)$?

Will this be the same as $\mathcal{L}\left(f(t)\cdot g(t)\right)$?

$$\frac{1}{(s-1)(s+1)} = \mathcal{L}\left(e^{t}\right) \cdot \mathcal{L}\left(e^{-t}\right)$$

Let $f(t) = e^{t}$ and $g(t) = e^{-t}$

$$\frac{1}{(s-1)(s+1)} = \mathcal{L}\left(f(t)\right) \dot{\mathcal{L}}\left(g(t)\right)$$

Compare this to $\mathcal{L}\left(f(t) \cdot g(t)\right)$

$$\mathcal{L}\left(f(t) \cdot g(t)\right) = \mathcal{L}\left(e^{t} \cdot e^{-t}\right) = \mathcal{L}\left(1\right) = \frac{1}{s}$$

Conclusion;

$$\mathcal{L}\left(f(t)\right) \cdot \mathcal{L}\left(g(t)\right) \neq \mathcal{L}\left(f(t) \cdot g(t)\right)$$

$$\int_0^1 u^3 \, du = \frac{1}{4}$$

$$\int_0^1 v^3 \, dv = \frac{1}{4}$$

$$\int_0^1 t^3 \, dt = \frac{1}{4}$$

$$\mathcal{L}\left(e^{t}\right) = \int_{0}^{\infty} e^{t} e^{-st}\, dt = \frac{1}{s-1}$$

$$\int_{0}^{\infty} e^{u} e^{-su}\, du = \frac{1}{s-1}$$

$$\int_{0}^{\infty} e^{v} e^{-sv}\, dv = \frac{1}{s-1}$$

$$\left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right) = \int_a^b \int_c^d f(x)g(y)\,dy\,dx$$

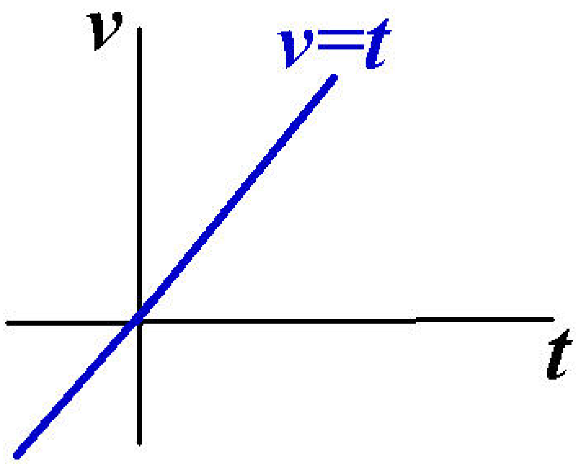
$$
\begin{aligned}
\mathcal{L}\left(f\right)\mathcal{L}\left(g\right) &= \int_0^\infty e^{-su} f(u)\,du \cdot \int_0^\infty e^{-sv} g(v)\,dv \\
&= \int_0^\infty \int_0^\infty e^{-su} f(u) e^{-sv} g(v)\,du\,dv \\
&= \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u) g(v)\,du\,dv
\end{aligned}
$$

$$\mathcal{L}\left(f\right)\mathcal{L}\left(g\right) = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u)g(v)\,du\,dv$$
$$= \int_0^\infty \left(\int_0^\infty e^{-s(u+v)} f(u)g(v)\,du\right)\,dv$$

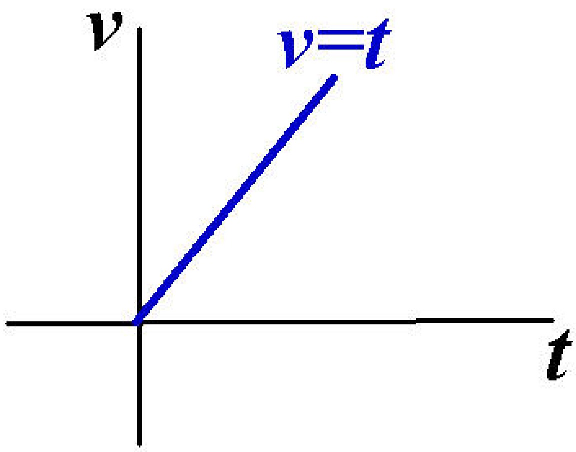
Let $t = u + v$ so $u = t - v$ and $du = dt$

$$\mathcal{L}\left(f\right)\mathcal{L}\left(g\right) = \int_0^\infty \left(\int_v^\infty e^{-st} f(t-v)g(v)\,dt\right)\,dv$$

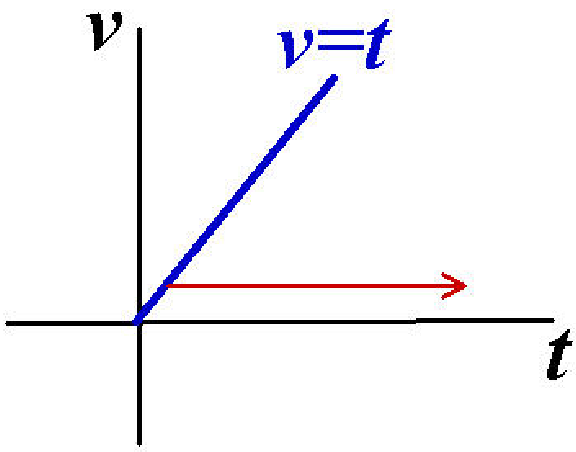
$$\mathcal{L}(f)\,\mathcal{L}(g) = \int_0^\infty \int_v^\infty e^{-st} f(t-v) g(v)\, dt\, dv$$



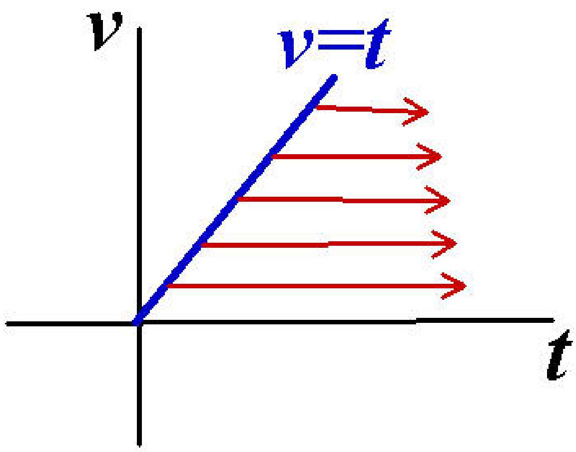
$$\mathcal{L}(f)\,\mathcal{L}(g) = \int_0^\infty \int_v^\infty e^{-st} f(t-v)g(v)\,dt\,dv$$



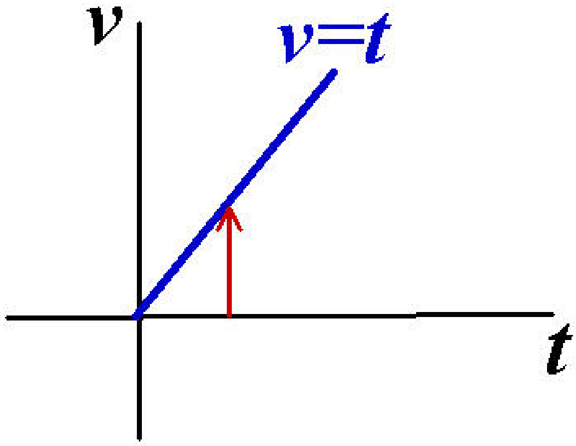
$$\mathcal{L}\left(f\right)\mathcal{L}\left(g\right) = \int_0^\infty \int_v^\infty e^{-st} f(t-v)g(v)\,dt\,dv$$



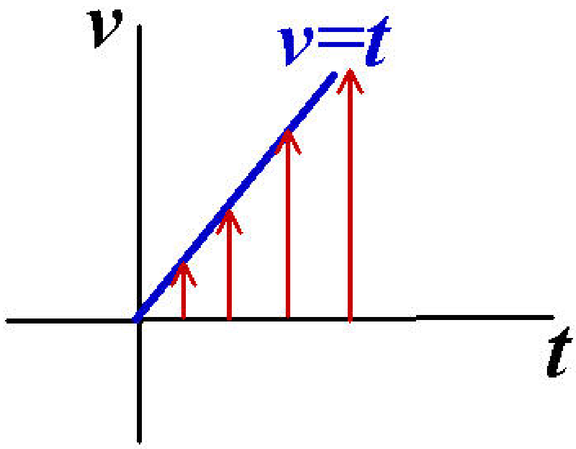
$$\mathcal{L}(f)\,\mathcal{L}(g) = \int_0^\infty \int_v^\infty e^{-st} f(t-v) g(v)\, dt\, dv$$



$$\mathcal{L}\,(f)\,\mathcal{L}\,(g) = \int_{?}^{?} \int_0^t e^{-st} f(t-v)g(v)\,dv\,dt$$



$$\mathcal{L}(f)\,\mathcal{L}(g) = \int_0^\infty \int_0^t e^{-st} f(t-v)g(v)\,dv\,dt$$



$$\begin{aligned}\mathcal{L}\left(f\right)\mathcal{L}\left(g\right) &= \int_0^\infty \int_0^t e^{-st} f(t-v)g(v)\,dv\,dt \\ &= \int_0^\infty e^{-st} \left(\int_0^t f(t-v)g(v)\,dv\right)\,dt\end{aligned}$$

$$\mathcal{L}\left(\text{Some function}\right) = \int_0^\infty e^{-st}(\text{Some function})\,dt$$

$$
\begin{aligned}
\mathcal{L}\left(f\right)\mathcal{L}\left(g\right) &= \int_0^\infty \int_0^t e^{-st} f(t-v)g(v)\,dv\,dt \\
&= \int_0^\infty e^{-st}\left(\int_0^t f(t-v)g(v)\,dv\right) \\
&= \mathcal{L}\left(\int_0^t f(t-v)g(v)\,dv\right)
\end{aligned}
$$

Definition: The Convolution Product:

$$(f * g)(t) = \int_0^t f(t - v)g(v)\, dv$$

The Convolution Theorem:

$$\mathcal{L}\left(f(t)\right)\mathcal{L}\left(g(t)\right) = \mathcal{L}\left((f * g)(t)\right)$$

$$y'' - y = 0 \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

$$\begin{aligned} \mathcal{L}(y) &= \frac{1}{s-1} \cdot \frac{1}{s+1} \\ &= \mathcal{L}\left(e^{t}\right) \cdot \mathcal{L}\left(e^{-t}\right) \end{aligned}$$

Let $f(t) = e^t$ and $g(t) = e^{-t}$

$\mathcal{L}(f(t))\,\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(t-v)g(v)\,dv\right)$

$$= \mathcal{L}\left(\int_0^t e^{t-v}e^{-v}\,dv\right) = \mathcal{L}\left(\int_0^t e^{t}e^{-2v}\,dv\right)$$

$$y'' - y = 0 \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

$$\begin{aligned}
\mathcal{L}\left(y\right) &= \frac{1}{s-1} \cdot \frac{1}{s+1} \\
&= \mathcal{L}\left(e^{t}\right) \cdot \mathcal{L}\left(e^{-t}\right) \\
&= \mathcal{L}\left(\int_0^t e^{t} e^{-2v}\, dv\right)
\end{aligned}$$

$$y'' - y = 0 \qquad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

$$\mathcal{L}\left(y\right) = \mathcal{L}\left(\int_0^t e^t e^{-2v}\, dv\right)$$

$$y = \int_0^t e^t e^{-2v}\, dv$$

$$
\begin{aligned}
y &= \int_0^t e^t e^{-2v}\, dv \\
&= e^t \int_0^t e^{-2v}\, dv \\
&= e^t \left[-\frac{1}{2} e^{-2v} \right]_0^t \\
&= e^t \left(\frac{1}{2} e^0 - \frac{1}{2} e^{-2t} \right) \\
&= \frac{1}{2} \left(e^t - e^{-t} \right)
\end{aligned}
$$

$$\mathcal{L}\left(f(t)\right)\mathcal{L}\left(g(t)\right) = \mathcal{L}\left(\int_0^t f(t-v)g(v)\,dv\right)$$

$$\mathcal{L}\left(g(t)\right)\mathcal{L}\left(f(t)\right) = \mathcal{L}\left(\int_0^t g(t-v)f(v)\,dv\right)$$

But $\mathcal{L}\left(f(t)\right)\mathcal{L}\left(g(t)\right) = \mathcal{L}\left(g(t)\right)\mathcal{L}\left(f(t)\right)$ so

$$\int_0^t f(t-v)g(v)\,dv = \int_0^t g(t-v)f(v)\,dv$$

$$(f * g)(t) = (g * f)(t)$$

$$y'' - y = 1 - t \qquad y(0) = y'(0) = 0$$

$$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(1) - \mathcal{L}(t)$$

$$s^2\mathcal{L}(y) - \mathcal{L}(y) = \frac{1}{s} - \frac{1}{s^2}$$

$$\mathcal{L}(y) = \frac{1}{(s+1)s^2} = \mathcal{L}\left(e^{-t}\right)\mathcal{L}(t)$$

$$\mathcal{L}(y) = \frac{1}{(s+1)s^2} = \mathcal{L}\left(e^{-t}\right)\mathcal{L}(t)$$

Let $f(t) = e^{-}t$ and $g(t) = t$
and use $\mathcal{L}(f(t))\,\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(t-v)g(v)\,dv\right)$

$$\mathcal{L}(y) = \mathcal{L}\left(\int_0^t e^{-(t-v)}v\,dv\right)$$

$$
\begin{aligned}
y &= \int_0^t e^{-(t-v)} v \, dv \\
&= e^{-t} \int_0^t e^{v} v \, dv \\
&= t - 1 + e^{-t}
\end{aligned}
$$

$$y'' + y = f(t) \quad \text{where } y(0) = y'(0) = 0$$

$$y'' + y = f(t) \quad \text{where } y(0) = y'(0) = 0$$

$$s^2 \mathcal{L}(y) + \mathcal{L}(y) = \mathcal{L}(f(t))$$

$$\mathcal{L}(y) = \frac{1}{s^2 + 1} \mathcal{L}(f(t))$$

$$y'' + y = f(t) \quad \text{where } y(0) = y'(0) = 0$$

$$s^2\mathcal{L}\left(y\right) + \mathcal{L}\left(y\right) = \mathcal{L}\left(f(t)\right)$$

$$\mathcal{L}\left(y\right) = \frac{1}{s^2+1}\mathcal{L}\left(f(t)\right)$$

$$\begin{aligned}\mathcal{L}\left(y\right) &= \mathcal{L}\left(\sin t\right)\mathcal{L}\left(f(t)\right) \\ &= \mathcal{L}\left(\int_0^t \sin(t-v)f(v)\,dv\right)\end{aligned}$$

$$\mathcal{L}\left(y\right) = \mathcal{L}\left(\int_0^t \sin(t-v)f(v)\,dv\right)$$

$$y = \int_0^t \sin(t-v)f(v)\,dv$$