Differential Equations
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Today's Topic: Series Solutions

$$y'' - 2y' = 0$$

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$$\text{Substitute} \quad e^{rx} : \quad r^2 e^{rx} - 2re^{rx} = 0$$

$$r^2 - 2r = 0$$

$$\text{Solutions:} \quad r = 0 \text{ and} \quad r = 2$$

$$y = ae^{0x} + be^{2x} = a + be^{2x}$$

$$y'' - 2xy' = 0$$

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What happens if we try substituting e^{rx}?

$$r^2 e^{rx} - 2xre^{rx} = 0$$

$$r^2 - 2xr = 0$$

$$r(r - 2x) = 0$$

$$r = 2x \quad \text{is a solution}$$

Contradiction!

The equation $y'' - 2xy' = 0$ does not have two linearly independent solutions of the form e^{rx}. In general, if the coefficients of a differential equation have variables in them, we may not get *any* solutions of the form e^{rx}.

$$y'' - 2xy' = 0$$

$$\frac{y''}{y'} = 2x$$

$$\frac{d}{dx}(\ln f(x)) = \frac{1}{f(x)} \cdot f'(x)$$

Therefore,

$$\frac{d}{dx}(\ln(y')) = \frac{1}{y'} \cdot y'' = \frac{y''}{y'}$$

$$y'' - 2xy' = 0$$

$$\frac{y''}{y'} = 2x$$

$$\frac{d}{dx}(\ln(y')) = 2x$$

$$\ln(y') = x^2 + C$$

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$$y' = e^{x^2+C} = e^C e^{x^2} = ke^{x^2} \quad \text{where} \quad k = e^C$$

$$y' = ke^{x^2}$$

$$y = k \int e^{x^2} \, dx$$

Power Series for $f(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\text{if} \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{then} \quad e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(x^2\right)^n$$

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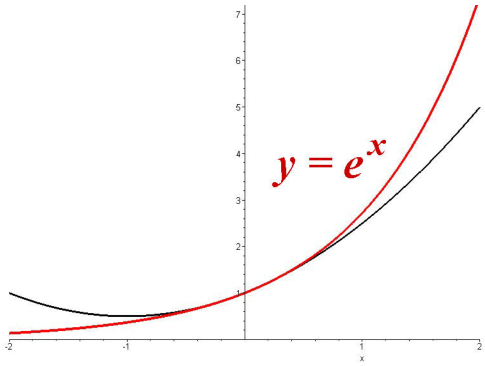
$$y = k \int e^{x^2} \, dx = k \int \left(1 + \frac{x^2}{1} + \frac{x^4}{2} + \cdots\right) dx$$

$$
\begin{aligned}
y &= k \int e^{x^2} dx \\
&= k \int \left(1 + \frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3!} + \cdots\right) dx \\
&= k \left(x + \frac{x^3}{3} + \frac{x^5}{(2)(5)} + \frac{x^7}{(3!)(7)} + \cdots\right)
\end{aligned}
$$

How does this help?

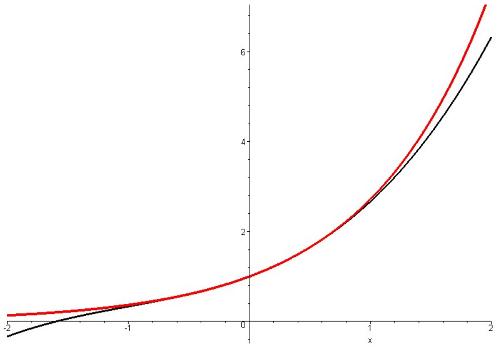
$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

Graph e^x and $1 + x + \frac{1}{2}x^2$ on the same axis.



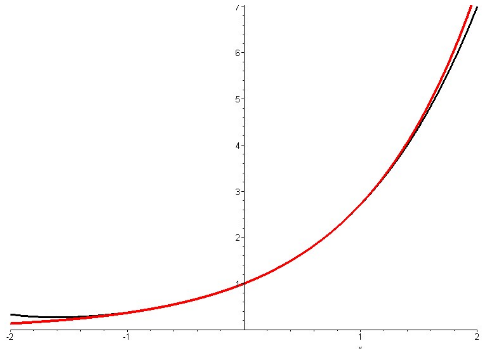
$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

Graph e^x and $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$ on the same axis.



Now the approximation is even better.

Let's add another term: $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$



If we are trying to solve a differential equation but all we knew about the solution was it's power series $\sum a_n x^n$, we could still obtain polynomial approximations for the solution.

How do we find the power series $\sum a_n x^n$ for a function $y = y(x)$ if we know the differential equation that y satisfies?

$$\frac{dy}{dx} - 7y = 0 \qquad \text{where } y(0) = 1$$

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If $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ then $y(0) = a_0$. Since the initial condition says that $y(0) = 1$, this is what a_0 is.

$$y = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\frac{dy}{dx} - 7y = 0 \qquad \text{where } y(0) = 1$$

If $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ then $y(0) = a_0$. Since the initial condition says that $y(0) = 1$, this is what a_0 is.

$$y = 1 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

One coefficient down, an infinite number to go...

If $y = 1 + a_1x + a_2x^2 + a_3x^3 + \cdots$ then $\frac{dy}{dx}$ is:

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

Now, back to the differential equation:

$$\frac{dy}{dx} - 7y = 0$$

$$(a_1 + 2a_2x + 3a_3x^2 + \cdots) - 7(1 + a_1x + a_2x^2 + \cdots) = 0$$

$$\frac{dy}{dx} - 7y = 0$$

$$(a_1 + 2a_2x + 3a_3x^2 + \cdots) - 7(1 + a_1x + a_2x^2 + \cdots) = 0$$

Now, combine and collect like terms:

$$a_1 - 7 + (2a_2 - 7a_1)x + (3a_3 - 7a_2)x^2 + (4a_4 - 7a_3)x^3 +$$
$$= 0$$

The only way this polynomial is 0 for all x is for each coefficient to be 0.

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More generally,

$$(n+1)a_{n+1} - 7a_n = 0$$

$$a_1 - 7 = 0 \qquad 2a_2 - 7a_1 = 0 \qquad 3a_3 - 7a_2 = 0 \qquad 4a_4 - 7a_3$$

$$\begin{aligned}
a_1 &= 7 \\
a_2 &= \frac{7a_1}{2} = \frac{7^2}{2} \\
a_3 &= \frac{7a_2}{3} = \frac{7^3}{(3)(2)} \\
a_4 &= \frac{7a_3}{4} = \frac{7^4}{(4)(3)(2)}
\end{aligned}$$

$$a_1 - 7 = 0 \qquad 2a_2 - 7a_1 = 0 \qquad 3a_3 - 7a_2 = 0 \qquad 4a_4 - 7a_3$$

$$a_1 = 7 = \frac{7^1}{1!}$$

$$a_2 = \frac{7a_1}{2} = \frac{7^2}{2} = \frac{7^2}{2!}$$

$$a_3 = \frac{7a_2}{3} = \frac{7^3}{(3)(2)} = \frac{7^3}{3!}$$

$$a_4 = \frac{7a_3}{4} = \frac{7^4}{(4)(3)(2)} = \frac{7^4}{4!}$$

$$
\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= 1 + \frac{7^1}{1!} x^1 + \frac{7^2}{2!} x^2 + \frac{7^3}{3!} x^3 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{7^n}{n!} x^n
\end{aligned}
$$

Series for e^{x^2}

$$1 + \frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3!} + \cdots$$

In $\sum$ notation, this would be expressed as:

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3!} + \cdots$$

Any of the following would mean the same thing:

$$\sum_{n=-1}^{\infty} \frac{x^{2(n+1)}}{(n+1)!} \qquad \sum_{n=-2}^{\infty} \frac{x^{2(n+2)}}{(n+2)!} \qquad \sum_{n=-3}^{\infty} \frac{x^{2(n+3)}}{(n+3)!}$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Any of the following would mean the same thing:

$$\sum_{n=0}^{\infty} a_n x^n \qquad \sum_{n=-1}^{\infty} a_{n+1} x^{n+1} \qquad \sum_{n=-2}^{\infty} a_{n+2} x^{n+2}$$

Define $a_n = 0$ when $n < 0$

$$a_{-1} = 0$$

$$a_{-2} = 0$$

$$a_{-3} = 0$$

$$a_{-4} = 0$$

$$\vdots$$

Define $a_n = 0$ when $n < 0$

$$\sum_{n=-\infty}^{\infty} a_n x^n = \cdots + a_{-3}x^{-3} + a_{-2}x^{-2} + a_{-1}x^{-1} + a_0 + a$$

$$= \cdots + 0x^{-3} + 0x^{-2} + 0x^{-1} + a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= \qquad a_0 + a_1 x + a_2 x^2 + \cdots$$

Abbreviation: $\sum a_n x^n$ means that n ranges from $-\infty$ to ∞. All of the following mean the same thing:

$$\sum a_n x^n \qquad \sum a_{n+1} x^{n+1}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Any of the following sums would equal $\frac{dy}{dx}$

$$\sum n a_n x^{n-1} \qquad \sum (n+1) a_{n+1} x^n \qquad \sum (n+2) a_{n+2} x^{n+1}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$y'' = 2a_2 + (3)(2)a_3 x + (4)(3)a_4 x^2 + (5)(4)a_5 x^3 + \cdots$$

Any of the following sums would equal y''

$$\sum n(n-1)a_n x^{n-2} \qquad \sum (n+1)n a_{n+1} x^{n-1}$$

$$\sum (n+2)(n+1)a_{n+2} x^n \qquad \sum (n+3)(n+2)a_{n+3} x^{n+1}$$

$$y' - 7y = 0$$

If we substitute $\sum a_n x^n$ in place of y, we get:

$$y' - 7\sum a_n x^n = 0$$

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We have several choices to use for y'

$$\sum n a_n x^{n-1} = \sum (n+1) a_{n+1} x^n = \sum (n+2) a_{n+2} x^{n+1}$$

$$\sum(n+1)a_{n+1}x^n - \sum 7a_nx^n = 0$$

$$\sum\left((n+1)a_{n+1} - 7a_n\right)x^n = 0$$

The only way this polynomial will equal 0 for all x is for each coefficient to equal 0.

$$(n+1)a_{n+1} - 7a_n = 0$$

This is referred to as the *recurrence relation*. This enables us to get a_{n+1} as soon as we know a_n.

Obtain a series of the form $\sum a_n x^n$ that solves:

$$y'' - y = 0 \quad \text{where } y(0) = 1 \text{ and } y'(0) = 1$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + \cdots = a_0$$

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$y'(0) = a_1 + 2a_2 \cdot 0 + 3a_3 \cdot 0 + \cdots = a_1$$

Obtain a series of the form $\sum a_n x^n$ that solves:

$$y'' - y = 0 \quad \text{where } y(0) = 1 \text{ and } y'(0) = 1$$

We start off knowing that $a_0 = 1$ and $a_1 = 1$
Substitute $\sum a_n x^n$ in place of y in the equation.

$$y'' - \sum a_n x^n = 0$$

Obtain a series of the form $\sum a_n x^n$ that solves:

$$y'' - y = 0 \quad \text{where } y(0) = 1 \text{ and } y'(0) = 1$$

$$y'' - \sum a_n x^n = 0$$

Choices for y'':

$$\sum n(n-1)a_n x^{n-2} \qquad \sum (n+1)n a_{n+1} x^{n-1}$$

$$\sum (n+2)(n+1)a_{n+2} x^n \qquad \sum (n+3)(n+2)a_{n+3} x^{n+1}$$

$$y'' - \sum a_n x^n = 0$$

$$\sum (n+2)(n+1)a_{n+2}x^n - \sum a_n x^n = 0$$

$$\sum \left((n+2)(n+1)a_{n+2} - a_n\right) x^n = 0$$

$$y'' - \sum a_n x^n = 0$$

$$\sum (n+2)(n+1)a_{n+2}x^n - \sum a_n x^n = 0$$

$$\sum \left((n+2)(n+1)a_{n+2} - a_n\right) x^n = 0$$

Each coefficient must equal 0

$$(n+2)(n+1)a_{n+2} - a_n = 0 \quad \text{for all } n$$

Each coefficient must equal 0

$$(n+2)(n+1)a_{n+2} - a_n = 0 \quad \text{for all } n$$

For $n \geq 0$, we can solve for a_{n+2} in terms of a_n.

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}$$

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We already know $a_0 = 1$ and $a_1 = 1$.
If $n = 0$, we get $a_2 = \frac{1}{(2)(1)}$

If $n = 1$, we get $a_3 = \frac{a_1}{(3)(2)} = \frac{1}{(3)(2)}$

If $n = 2$, we get $a_4 = \frac{a_2}{(4)(3)} = \frac{1}{(4)(3)(2)(1)}$

If $n = 3$, we get $a_5 = \frac{a_3}{(5)(4)} = \frac{1}{(5)(4)(3)(2)(1)}$

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)}$$

We already know $a_0 = 1$ and $a_1 = 1$.
If $n = 0$, we get $a_2 = \frac{1}{(2)(1)} = \frac{1}{2!}$

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If $n = 2$, we get $a_4 = \frac{a_2}{(4)(3)} = \frac{1}{(4)(3)(2)(1)} = \frac{1}{4!}$

If $n = 3$, we get $a_5 = \frac{a_3}{(5)(4)} = \frac{1}{(5)(4)(3)(2)(1)} = \frac{1}{5!}$

$$a_n = \frac{1}{n!}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

We are ready to solve the *variable coefficient* case.