Differential Equations
Dr. E. Jacobs

Today's Topic: The Variable Coefficient Problem

Equations with Constant Coefficients

$$y'' + 2y' + 2y = 0$$

$$y'' + 16y' + 8y = 0$$

Equations with Variable Coefficients

$$y'' + xy' + 2y = 0$$

$$\left(1 + 4x^2\right) y'' + 16xy' + 8y = 0$$

Series solutions -
Solve for the coefficients a_0, a_1, a_2, ..., a_n, ... so that the following series is the solution to a given differential equation.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = \sum a_n x^n$$

The following also mean the same thing:

$$\sum a_{n+1} x^{n+1} \qquad \sum a_{n+2} x^{n+2}$$

$$y = \sum a_n x^n$$

Also: $\sum a_{n+1} x^{n+1}$ $\sum a_{n+2} x^{n+2}$

$$y' = \sum n a_n x^{n-1}$$

Also: $\sum (n+1) a_{n+1} x^n$ $\sum (n+2) a_{n+2} x^{n+1}$

$$y' = \sum n a_n x^{n-1}$$

Also: $\sum (n+1) a_{n+1} x^n$ $\qquad$ $\sum (n+2) a_{n+2} x^{n+1}$

$$y'' = \sum n(n-1) a_n x^{n-2}$$

Also:
$\sum (n+1_{)} n a_{n+1} x^{n-1}$ $\quad$ $\sum (n+2)(n+1) a_{n+2} x^n$

Example:

$$y'' + xy' + 2y = 0 \quad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

$a_0 = y(0) = 0$ and $a_1 = y'(0) = 1$
$y = \sum a_n x^n$ and $y'' = \sum (n+2)(n+1)a_{n+2}x^n$

$$\begin{aligned} y' &= \sum n a_n x^{n-1} = \sum (n+1)a_{n+1}x^n \\ &= \sum (n+2)a_{n+2}x^{n+1} \end{aligned}$$

Example:

$$y'' + xy' + 2y = 0 \quad \text{where } y(0) = 0 \text{ and } y'(0) = 1$$

$a_0 = y(0) = 0$ and $s_1 = y'(0) = 1$
$y = \sum a_n x^n$ and $y'' = \sum (n+2)(n+1)a_{n+2}x^n$

Choose $$y' = \sum n a_n x^{n-1}$$

$$xy' = x \sum n a_n x^{n-1} = \sum n a_n x^n$$

$$y'' + xy' + 2y = 0$$

$$\sum (n+2)(n+1)a_{n+2}x^n + \sum na_nx^n + 2\sum a_nx^n = 0$$

$$\sum \left((n+2)(n+1)a_{n+2} + na_n + 2a_n\right)x^n = 0$$

This can only happen if the coefficient of x^n is 0 for each n.

$$(n+2)(n+1)a_{n+2} + (n+2)a_n = 0$$

$$(n+2)(n+1)a_{n+2} + (n+2)a_n = 0$$

For $n \geq 0$ we get:

$$a_{n+2} = \frac{-a_n}{n+1}$$

For $n = 0$, $a_2 = \frac{-a_0}{0+1} = 0$ For $n = 1$ $a_3 = \frac{-a_1}{2} = -\frac{1}{2}$

For $n = 2$, $a_4 = \frac{-a_2}{2+1} = 0$ For $n = 3$ $a_5 = \frac{-a_3}{4} = \frac{1}{(4)(2)}$

$$a_{n+2} = \frac{-a_n}{n+1}$$

For $n = 0$, $a_2 = \frac{-a_0}{0+1} = 0$ For $n = 1$ $a_3 = \frac{-a_1}{2} = -\frac{1}{2}$

For $n = 2$, $a_4 = \frac{-a_2}{2+1} = 0$ For $n = 3$ $a_5 = \frac{-a_3}{4} = \frac{1}{(4)(2)}$

$$a_6 = 0 \qquad a_8 = 0 \qquad a_{10} = 0 \cdots$$

$$a_{n+2} = \frac{-a_n}{n+1}$$

For $n = 1$, $a_3 = \frac{-a_1}{2} = -\frac{1}{2}$

For $n = 3$, $a_5 = \frac{-a_3}{4} = \frac{1}{(4)(2)}$

For $n = 5$, $a_7 = \frac{-a_5}{6} = -\frac{1}{(6)(4)(2)}$

For $n = 7$, $a_9 = \frac{-a_7}{8} = \frac{1}{(8)(6)(4)(2)}$

$$a_{n+2} = \frac{-a_n}{n+1}$$

For $n = 1$, $a_3 = \frac{-a_1}{2} = -\frac{1}{2} = -\frac{1}{2^1 1!}$

For $n = 3$, $a_5 = \frac{-a_3}{4} = \frac{1}{(4)(2)} = \frac{1}{2^2 2!}$

For $n = 5$, $a_7 = \frac{-a_5}{6} = -\frac{1}{(6)(4)(2)} = -\frac{1}{2^3 3!}$

For $n = 7$, $a_9 = \frac{-a_7}{8} = \frac{1}{(8)(6)(4)(2}) = \frac{1}{2^4 4!}$

$$
\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \\
&= a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 \cdots
\end{aligned}
$$

$$
\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \\
&= a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 \cdots \\
&= x - \frac{1}{2^1 1!} x^3 + \frac{1}{2^2 2!} x^5 - \frac{1}{2^3 3!} x^7 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \\
&= a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 \cdots \\
&= x - \frac{1}{2^1 1!} x^3 + \frac{1}{2^2 2!} x^5 - \frac{1}{2^3 3!} x^7 + \cdots \\
&= x \left(1 - \frac{1}{2^1 1!} x^2 + \frac{1}{2^2 2!} x^4 - \frac{1}{2^3 3!} x^6 + \cdots \right) \\
&= x \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n}
\end{aligned}
$$

Example:

Find a solution of the form $\sum a_n x^n$ that solves:

$$\left(1+4x^2\right)y'' + 16xy' + 8y = 0$$

where $y(0) = 1, \; y'(0) = 0$

Note that $a_0 = y(0) = 1$ and $a_1 = y'(0) = 0$

$$\left(1+4x^2\right)y''+16xy'+8y=0$$

$$y''+4x^2y''+16xy'+8y=0$$

$$y''+4x^2y''+16xy'+8\sum a_nx^n=0$$

Choices for y' :

$\sum na_nx^{n-1}$ $\qquad$ $\sum(n{+}1)a_{n+1}x^n$ $\qquad$ $\sum(n{+}2)a_{n+2}x^{n+1}$

Pick the first one

$$y'' + 4x^2y'' + 16xy' + 8y = 0$$

$$y'' + 4x^2y'' + 16\sum na_nx^n + 8\sum a_nx^n = 0$$

Choices for y'' :

$$\sum n(n-1)a_nx^{n-2} \quad \sum(n+1)na_{n+1}x^{n-1}$$

$$\sum(n+2)(n+1)a_{n+2}x^n \quad \sum(n+3)(n+2)a_{n+3}x^{n+1}$$

$$y'' + 4x^2y'' + 16xy' + 8y = 0$$

$$y'' + 4x^2y'' + 16\sum na_nx^n + 8\sum a_nx^n = 0$$

Pick $\sum(n+2)(n+1)a_{n+2}x^n$ for the first term
Pick $\sum n(n-1)a_nx^{n-2}$ for the second term

$$y'' + 4x^2y'' + 16xy' + 8y = 0$$

$$\sum (n+2)(n+1)a_{n+2}x^n + \sum 4n(n-1)a_nx^n + \sum 16na_nx^n + \sum 8a_nx^n = 0$$

$$\sum \left((n+2)(n+1)a_{n+2} + (4n(n-1) + 16n + 8)a_n\right)x^n = 0$$

Each coefficient must be 0 so:

$$(n+2)(n+1)a_{n+2} + (4n(n-1) + 16n + 8)a_n = 0$$

$$(n+2)(n+1)a_{n+2} + (4n(n-1) + 16n + 8)a_n = 0$$

The equation simplifies:

$$(n+2)(n+1)a_{n+2} + (4n^2 + 12n + 8)a_n = 0$$

$$(n+2)(n+1)a_{n+2} + 4(n+2)(n+1)a_n = 0$$

$$a_{n+2} = -4a_n$$

$$a_{n+2} = -4a_n$$

We already know that $a_0 = 1$ and $a_1 = 0$

For $n = 0$, $a_2 = -4a_0 = -4$
For $n = 1$, $a_3 = -4a_1 = 0$
For $n = 2$, $a_4 = -4a_2 = 4^2$
For $n = 3$, $a_5 = -4a_3 = 0$

$$a_{n+2} = -4a_n$$

We already know that $a_0 = 1$ and $a_1 = 0$

For $n = 0$, $a_2 = -4a_0 = -4$
For $n = 1$, $a_3 = -4a_1 = 0$
For $n = 2$, $a_4 = -4a_2 = 4^2$
For $n = 3$, $a_5 = -4a_3 = 0$

$$a_7 = 0 \qquad a_9 = 0 \qquad a_{11} = 0 \qquad \cdots$$

$$a_6 = -4^3 \qquad a_8 = 4^4 \qquad a_{10} = -4^5 \qquad \cdots$$

$$
\begin{aligned}
y &= a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots \\
&= 1 - 4x^2 + 4^2 x^4 - 4^3 x^6 + \cdots
\end{aligned}
$$

It's always nice if you can write the answer in summation form:

$$
y = \sum_{n=0}^{\infty} (-4)^n x^{2n}
$$

$$y = \sum_{n=0}^{\infty} (-4)^n x^{2n}$$

$$1+r+r^2+r^3+r^4+\cdots = \frac{1}{1-r} \qquad \text{as long as } |r| < 1$$

In summation form:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$y = \sum_{n=0}^{\infty} (-4)^n x^{2n} = \sum_{n=0}^{\infty} \left(-4x^2\right)^n$$

$$1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r} \qquad \text{as long as } |r| < 1$$

In summation form:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

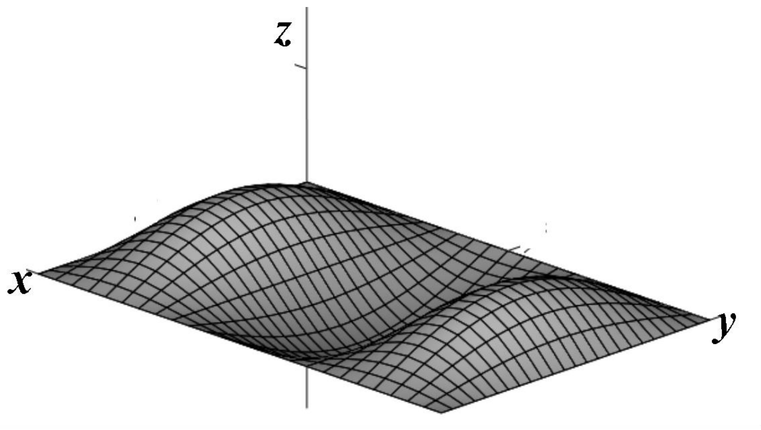
$$y = \sum_{n=0}^{\infty} \left(-4x^2\right)^n = \frac{1}{1-(-4x^2)} = \frac{1}{1+4x^2}$$

$r = -4x^2$. The series only converges if $|r| < 1$ so:

$$4x^2 < 1$$

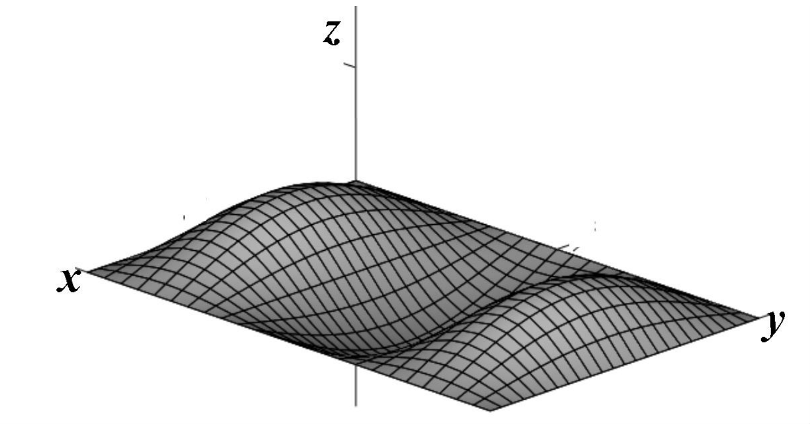
$$-\frac{1}{2} < x < \frac{1}{2}$$

Let $u = u(x, y, t)$ be the amplitude of a wave



Let $u = u(x, y, t)$ be the amplitude of a wave

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$



Vibration of a drumhead:

$$u = u(r,\, \theta,\, t)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$