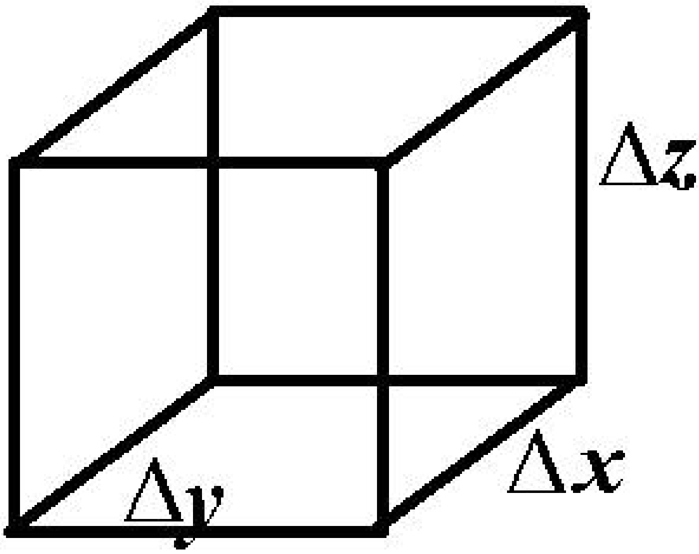
The Time Rate of Change of Density

Let's consider a more general cube with sides of length Δx, Δy and Δz and volume $\Delta V = \Delta x \Delta y \Delta z$



Let $M(t)$ denote mass inside the cube at time t. Since the flux through the surface S of the cube is the rate at which the mass is passing out of the surface then

$$\Phi_S = -\frac{dM}{dt}$$

Let $\rho = \rho(x, y, z, t)$ be the density (in kilograms per meter^3) at location (x, y, z) at time t seconds. If V denotes the interior of the solid then

$$M(t) = \iiint_V \rho \, dV$$

If $\mathrm{vol}(S) \approx 0$ and (x, y, z) is the center of the cube then the mass is approximated by:

$$M(t) = \iiint_V \rho \, dV \approx \rho \, \mathrm{vol}(V)$$

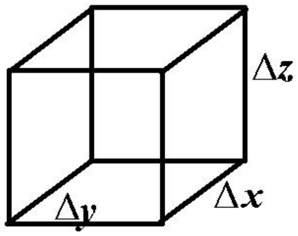
$$M(t) \approx \rho \, \mathrm{vol}(V)$$

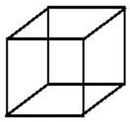
Therefore, the flux is approximated by:

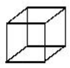
$$\Phi_S = -\frac{dM}{dt} \approx -\frac{\partial \rho}{\partial t} \, \mathrm{vol}(V)$$

$$\frac{\Phi_S}{\mathrm{vol}(V)} \approx -\frac{\partial \rho}{\partial t}$$

This approximation improves as $\mathrm{vol}(V) \longrightarrow 0$







In the limit,

$$\lim_{\mathrm{vol}(V)\to 0} \frac{\Phi_S}{\mathrm{vol}(V)} = -\frac{\partial \rho}{\partial t}$$

This is the **divergence** of the vector field $\vec{\mathbf{F}}$.

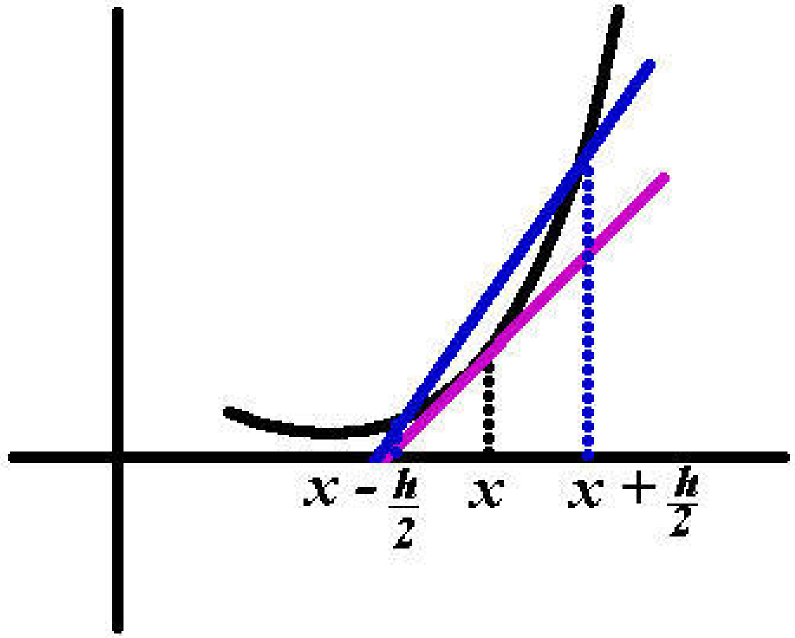
$$\operatorname{div} \vec{\mathbf{F}} = -\frac{\partial \rho}{\partial t}$$

Partial Derivative Formula for div $\vec{\mathbf{F}}$

Recall that the derivative of $f(x)$ is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}{h}$$



$$f'(x) = \lim_{h \to 0} \frac{f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}{h}$$

$$\frac{f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)}{h} \approx f'(x)$$

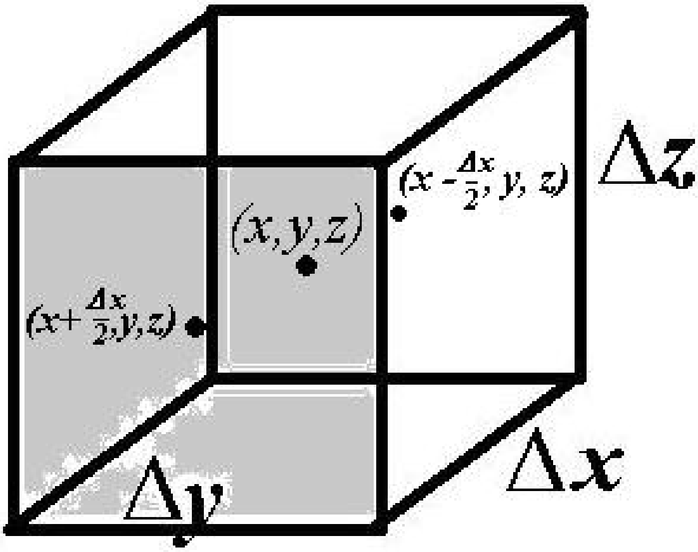
where the approximation improves as $h \to 0$

We can do exactly the same sort of thing for partial derivatives. If $f = f(x, y, z)$ then

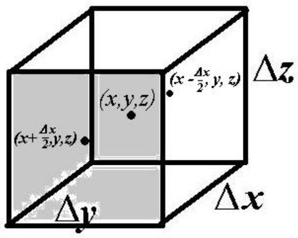
$$\frac{f\left(x + \frac{\Delta x}{2},\ y,\ z\right) - f\left(x - \frac{\Delta x}{2},\ y,\ z\right)}{\Delta x} \approx \frac{\partial f}{\partial x}$$

where the error in approximation $\to 0$ as $\Delta x \to 0$

Consider the flux through the following cube:



On the front portion, $\vec{\mathbf{n}} = \langle 1, 0, 0 \rangle$

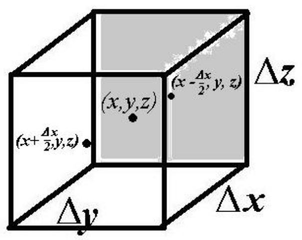


If this is a small cube, then the flux through the front may be approximated at $\left(x + \frac{\Delta x}{2},\ y,\ z\right)$

$$\begin{aligned}
\iint_{S_1} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \iint_{S_1} \langle F_1,\ F_2,\ F_3 \rangle \bullet \vec{\mathbf{i}}\, dy\, dz \\
&= \iint_{S_1} F_1\, dy\, dz \\
&\approx F_1 \left(x + \frac{\Delta x}{2},\ y,\ z \right) \Delta y\, \Delta z
\end{aligned}$$

Flux through the back:

$$\iint_{S_2} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{S_2} \langle F_1,\ F_2,\ F_3 \rangle \bullet (-\vec{\mathbf{i}})\, dy\, dz$$
$$= \iint_{S_2} -F_1\, dy\, dz$$
$$\approx -F_1\left(x - \frac{\Delta x}{2},\ y,\ z\right) \Delta y\, \Delta z$$



Total approximate flux through the front and back:

$$\left(F_1\left(x+\tfrac{\Delta x}{2},\ y,\ z\right)-F_1\left(x-\tfrac{\Delta x}{2},\ y,\ z\right)\right)\Delta y\,\Delta z$$

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This is equal to:

$$\frac{F_1\left(x+\tfrac{\Delta x}{2},\ y,\ z\right)-F_1\left(x-\tfrac{\Delta x}{2},\ y,\ z\right)}{\Delta x}\cdot\Delta x\,\Delta y\,\Delta z$$

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$$\frac{F_1\left(x+\frac{\Delta x}{2},\ y,\ z\right)-F_1\left(x-\frac{\Delta x}{2},\ y,\ z\right)}{\Delta x}\cdot\Delta x\,\Delta y\,\Delta z$$

This is approximately equal to:

$$\frac{\partial F_1}{\partial x}(x,y,z)\Delta x\,\Delta y\,\Delta z$$

Flux approximation through the front and back as:

$$\frac{\partial F_1}{\partial x}\mathrm{Vol}(V)$$

Flux approximation through the front and back as:

$$\frac{\partial F_1}{\partial x} \mathrm{Vol}(V)$$

Through left and right sides:

$$\frac{\partial F_2}{\partial y} \mathrm{Vol}(V)$$

Through top and bottom:

$$\frac{\partial F_3}{\partial z} \mathrm{Vol}(V)$$

If we combine these quantities, we get the approximation of the flux through the entire surface S surrounding the cube:

$$\Phi_S \approx \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) \mathrm{Vol}(V)$$

The approximation becomes better and better as the volume shrinks to 0. Consequently, the divergence is given by:

$$\mathrm{div}\, \vec{\mathbf{F}} = \lim_{Vol(V) \to 0} \frac{\Phi_S}{\mathrm{Vol}(V)} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Example:

Let $\vec{\mathbf{F}} = \langle xy,\ z^2 \sin x,\ e^z x \rangle$.

$$F_1 = xy \qquad\qquad F_2 = z^2 \sin x \qquad\qquad F_3 = e^z x$$

$$\operatorname{div} \vec{\mathbf{F}} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(z^2 \sin x\right) + \frac{\partial}{\partial z}\left(e^z x\right) = y + e^z x$$

Example:

$\vec{\mathbf{F}} = \langle e^{-x},\ 0,\ 0\rangle.$

$\text{div}\ \vec{\mathbf{F}} = \frac{\partial}{\partial x}\left(e^{-x}\right) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = -e^{-x}$

∇ Notation for Divergence

We can write $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ as a dot product:

$$\text{div } \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle \bullet \langle F_1,\ F_2,\ F_3 \rangle$$

$$\text{Define the notation:} \quad \nabla = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle$$

The divergence is therefore:

$$\text{div } \vec{\mathbf{F}} = \nabla \bullet \vec{\mathbf{F}}$$

Recall that the derivative $f'(x)$ can be written as Df. Compare the derivative of a function Df to the divergence $\nabla \bullet \vec{\mathbf{F}}$.

Product Rule:

$$D(fg) = fDg + gDf$$

If $\phi(x, y, z)$ is a scalar-valued function and $\vec{\mathbf{F}} = \langle F_1(x, y, z),\ F_2(x, y, z),\ F_3(x, y, z)\rangle$ is a vector field then $\phi\vec{\mathbf{F}}$ is also a vector field (a scalar times a vector) and its divergence is:

$$\operatorname{div}(\phi\vec{\mathbf{F}}) = \nabla \bullet (\phi\vec{\mathbf{F}})$$

$$
\begin{aligned}
\operatorname{div}(\phi\vec{\mathbf{F}}) &= \nabla \bullet (\phi\vec{\mathbf{F}}) \\
&= \frac{\partial}{\partial x}(\phi F_1) + \frac{\partial}{\partial y}(\phi F_2) + \frac{\partial}{\partial z}(\phi F_3) \\
&= \phi\frac{\partial F_1}{\partial x} + \frac{\partial \phi}{\partial x}F_1 + \phi\frac{\partial F_2}{\partial y} + \frac{\partial \phi}{\partial y}F_2 \\
&\qquad\qquad + \phi\frac{\partial F_3}{\partial z} + \frac{\partial \phi}{\partial z}F_3 \\
&= \phi\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \\
&\qquad\qquad + \frac{\partial \phi}{\partial x}F_1 + \frac{\partial \phi}{\partial y}F_2 + \frac{\partial \phi}{\partial z}F_3 \\
&= \phi\nabla \bullet \vec{\mathbf{F}} + \nabla\phi \bullet \vec{\mathbf{F}}
\end{aligned}
$$

Compare $D(fg) = fDg + gDf$ to

$$\nabla \bullet (\phi\vec{\mathbf{F}}) = \phi\,\nabla \bullet \vec{\mathbf{F}} + \nabla\phi \bullet \vec{\mathbf{F}}$$

$$\nabla \bullet (\phi \vec{\mathbf{F}}) = \phi \, \nabla \bullet \vec{\mathbf{F}} + \nabla \phi \bullet \vec{\mathbf{F}}$$

Special Case:

Suppose the scalar ϕ is a constant C.
$\nabla \phi = \left\langle \frac{\partial}{\partial x}(C),\ \frac{\partial}{\partial y}(C),\ \frac{\partial}{\partial x}(C), \right\rangle = \langle 0,\ 0,\ 0 \rangle = \vec{\mathbf{0}}$

$$\begin{aligned} \operatorname{div}(C\vec{\mathbf{F}}) &= C \, \nabla \bullet F + \nabla C \bullet \vec{\mathbf{F}} \\ &= C \, \nabla \bullet F + \vec{\mathbf{0}} \bullet \vec{\mathbf{F}} \\ &= C \, \nabla \bullet F \\ &= C \operatorname{div} \, \vec{\mathbf{F}} \end{aligned}$$

A fluid is *incompressible* if the density remains constant. If ρ is the constant density and $\vec{\mathbf{v}}$ is the velocity vector field of the fluid, and $\vec{\mathbf{F}} = \rho\vec{\mathbf{v}}$ then

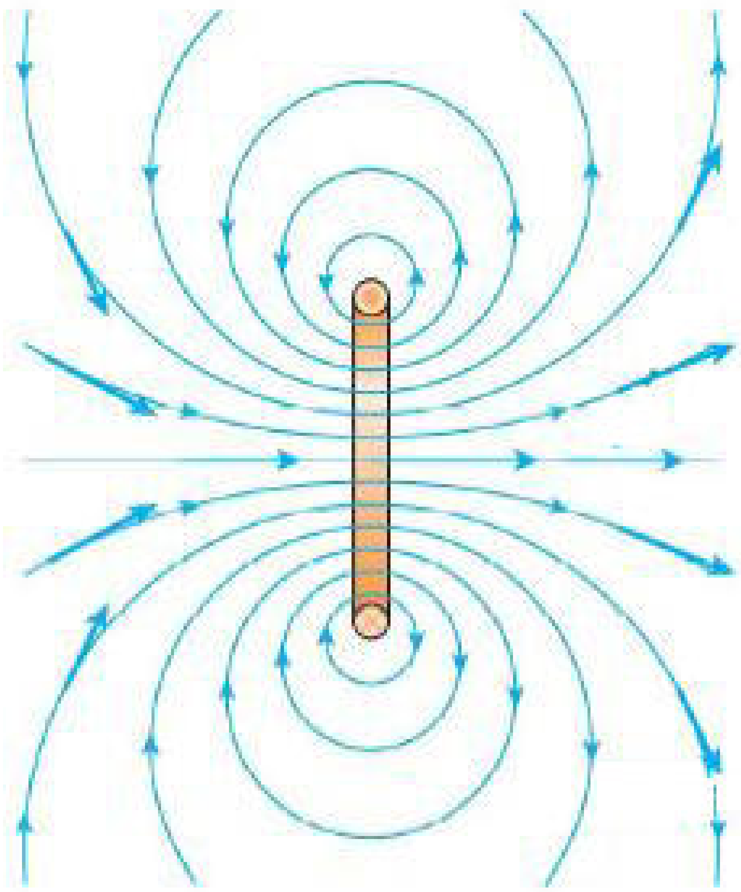
$$\text{div } \vec{\mathbf{F}} = -\frac{\partial \rho}{\partial t}$$

$$\text{div}(\rho\vec{\mathbf{v}}) = -\frac{\partial}{\partial t}(\text{constant}) = 0$$

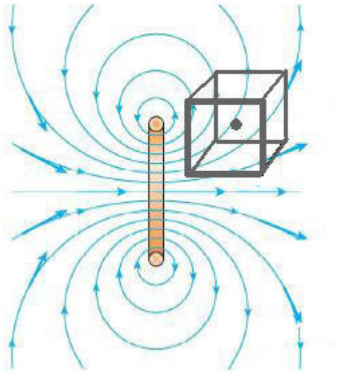
$$\rho \, \text{div } \vec{\mathbf{v}} = 0$$

Dividing both sides by ρ leaves us with:

$$\text{div} \, \vec{\mathbf{v}} = 0$$



Let's take some arbitrary point (x, y, z) and construct a closed surface around it.



Gauss's Law for Magnetism:

$$\Phi_S = \iint_S \vec{\mathbf{B}} \bullet \vec{\mathbf{n}} \, dS = 0$$

$$\Phi_S = \iint_S \vec{\mathbf{B}} \bullet \vec{\mathbf{n}}\, dS = 0$$

If flux $= 0$ then the flux per unit volume is also 0.

$$\frac{\Phi_S}{\mathrm{vol}(V)} = 0$$

Let $\mathrm{vol}(V) \to 0$

$$\nabla \bullet \vec{\mathbf{B}} = 0$$

$$\iint_S \vec{\mathbf{B}} \bullet \vec{\mathbf{n}}\, dS = 0$$

$$\nabla \bullet \vec{\mathbf{B}} = 0$$