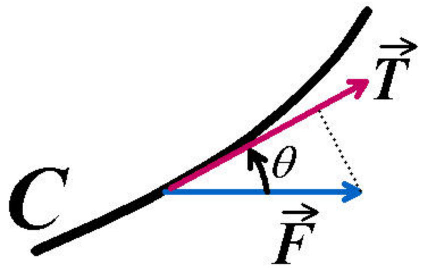
Work

Suppose a force $\vec{\mathbf{F}}$ is directed at an angle θ to the path C.
magnitude of tangential component $= |\vec{\mathbf{F}}| \cos \theta$



If $\vec{\mathbf{T}}$ is the unit tangent vector, then $|\vec{\mathbf{F}}| \cos\theta = \vec{\mathbf{F}} \bullet \vec{\mathbf{T}}$ and :

$$W = \int_C \vec{\mathbf{F}} \bullet \vec{\mathbf{T}} \, ds$$

$$\vec{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

$ds = |\vec{\mathbf{v}}|\, dt$ so:

$$\vec{\mathbf{T}}\, ds = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}\, |\vec{\mathbf{v}}|\, dt = \vec{\mathbf{v}}\, dt$$

If the initial point on the curve occurs at $t = a$ and the final point is at $t = b$, then work is given by:

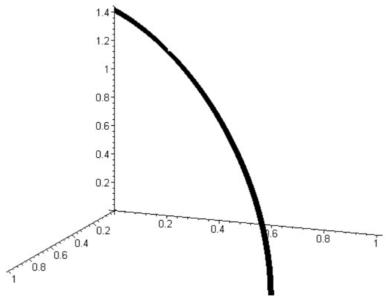
$$W = \int_a^b \vec{\mathbf{F}} \bullet \vec{\mathbf{v}}\, dt$$

Note that $\vec{\mathbf{v}}\, dt = \frac{d\vec{\mathbf{r}}}{dt}\, dt = d\vec{\mathbf{r}}$, so :

$$W = \int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

Example:

$$\vec{\mathbf{r}} = \langle x(t),\ y(t),\ z(t)\rangle = \langle \sin t,\ \sin t,\ \cos t\rangle \qquad 0 \le t \le \frac{\pi}{2}$$



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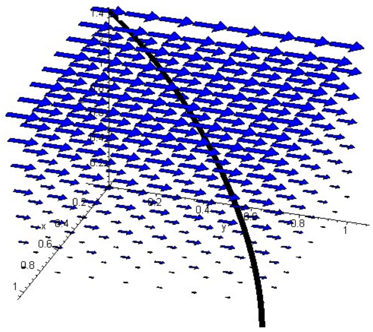
$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0\rangle = z\vec{\mathbf{j}}$$



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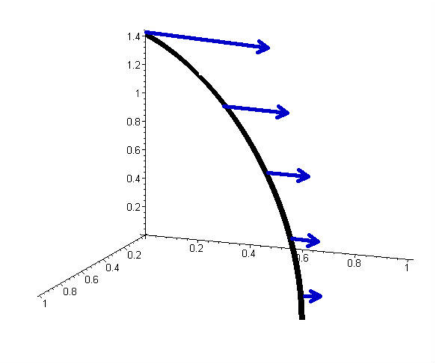

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$$\begin{aligned}\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} &= \int_0^{\pi/2} \langle 0,\ \cos t,\ 0\rangle \bullet \langle \cos t,\ \cos t,\ -\sin t\rangle\, dt \\ &= \int_0^{\pi/2} \cos^2 t\, dt = \frac{\pi}{4}\end{aligned}$$

What does it mean to reverse the limits of integration?

$$\int_0^{\pi/2} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = \frac{\pi}{4}$$

$$\int_{\pi/2}^{0} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = -\frac{\pi}{4}$$

Reversing the limits of integration changes the direction that the path is traversed.

$$\int_{\pi/2}^{0} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = -\frac{\pi}{4} \qquad \int_{0}^{\pi/2} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = \frac{\pi}{4}$$

$$\int_{-C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = -\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

Alternate method of calculation:

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_C \langle F_1,\ F_2,\ F_3 \rangle \bullet \langle dx,\ dy,\ dz \rangle = \int_C F_1\, dx + F_2\, dy + F_3\, dz$$

$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle \qquad d\vec{\mathbf{r}} = \langle dx,\ dy,\ dz \rangle$$

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} z\, dy$$

$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle \qquad d\vec{\mathbf{r}} = \langle dx,\ dy,\ dz \rangle$$

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For $\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle \sin t,\ \sin t,\ \cos t \rangle$:

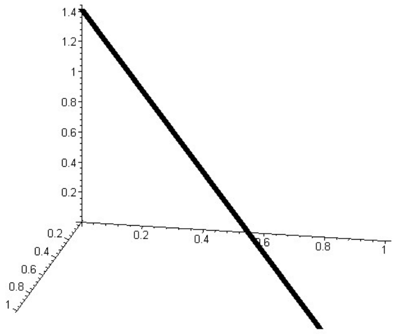
$$y^2 + z^2 = 1 \quad \text{so} \quad z = \sqrt{1 - y^2} \text{ so:}$$

$$\int_{C_1} z\, dy = \int_0^1 \sqrt{1 - y^2}\, dy$$

What happens if we change the path between (0, 0, 1) and (1, 1, 0)?

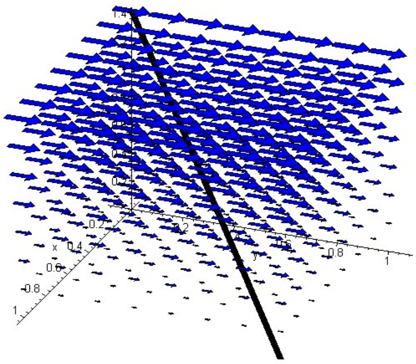
$$\vec{\mathbf{r}} = \langle 0,\ 0,\ 1\rangle + t\langle 1,\ 1,\ -1\rangle = \langle t,\ t,\ 1-t\rangle$$

$$0 \le t \le 1$$



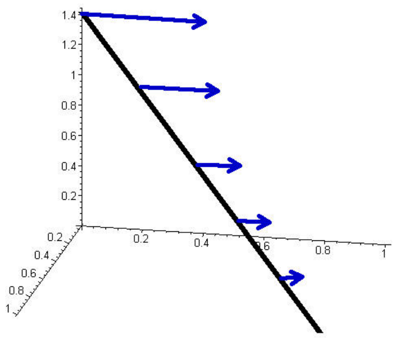
$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0\rangle$$

$$\vec{\mathbf{r}} = \langle 0,\ 0,\ 1\rangle + t\langle 1,\ 1,\ -1\rangle = \langle t,\ t,\ 1-t\rangle$$



$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle$$

$$\vec{\mathbf{r}} = \langle t,\ t,\ 1-t \rangle$$



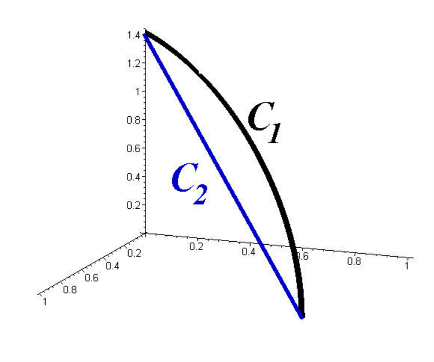
$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle = \langle 0,\ 1 - t,\ 0 \rangle$$

$$\vec{\mathbf{r}} = \langle t,\ t,\ 1 - t \rangle \qquad d\vec{\mathbf{r}} = \frac{d\vec{\mathbf{r}}}{dt}\, dt = \langle 1,\ 1,\ -1 \rangle\, dt$$

$$\int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^1 (1 - t)\, dt = \frac{1}{2}$$

For $\vec{\mathbf{F}} = z\vec{\mathbf{j}}$,

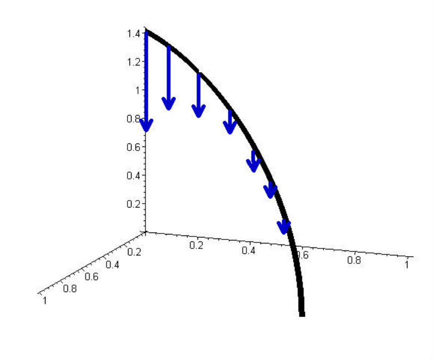
$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



Let's try this with a different vector field

$$\vec{\mathbf{F}} = -z\vec{\mathbf{k}} = \langle 0,\ 0,\ -z \rangle$$

$$\vec{\mathbf{r}} = \langle \sin t,\ \sin t,\ \cos t \rangle \qquad 0 \le t \le \frac{\pi}{2}$$



$$\vec{\mathbf{F}} = -z\vec{\mathbf{k}} = \langle 0,\ 0,\ -z\rangle = \langle 0,\ 0,\ -\cos t\rangle$$

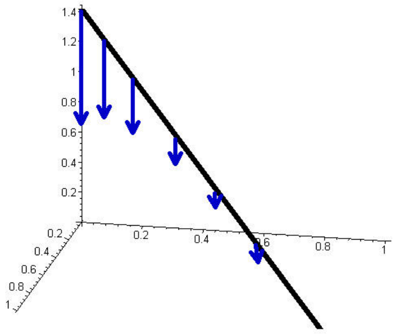
$$\vec{\mathbf{r}} = \langle \sin t,\ \sin t,\ \cos t\rangle \qquad 0 \le t \le \frac{\pi}{2}$$

$$d\vec{\mathbf{r}} = \frac{d\vec{\mathbf{r}}}{dt}\,dt = \langle \cos t,\ \cos t,\ -\sin t\rangle\,dt$$

$$\vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \sin t \cos t\,dt$$

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^{\pi/2} \sin t \cos t\,dt = \frac{1}{2}$$

Try $\vec{\mathbf{F}} = -z\vec{\mathbf{k}}$ along the straight line path C_2



$$\vec{\mathbf{r}} = \langle t,\ t,\ 1 - t\rangle \qquad d\vec{\mathbf{r}} = \langle 1,\ 1,\ -1\rangle\, dt$$

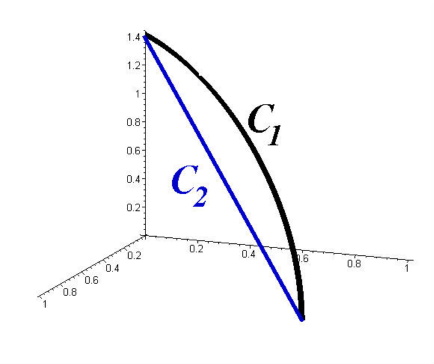
$$\vec{\mathbf{F}} = \langle 0,\ 0,\ -z\rangle = \langle 0,\ 0,\ -(1-t)\rangle$$

$$\vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = (1-t)\, dt$$

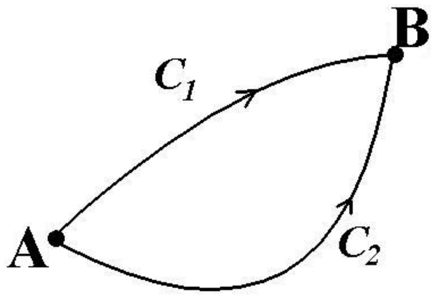
$$\int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^1 (1-t)\, dt = \frac{1}{2}$$

This time, for $\vec{\mathbf{F}} = -z\vec{\mathbf{k}}$, the path doesn't seem to matter.

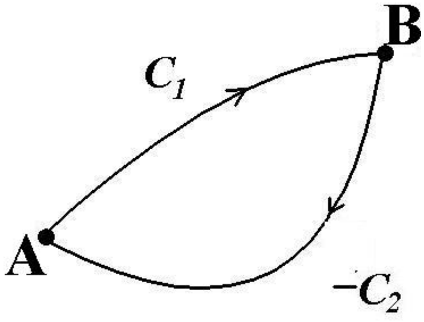
$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



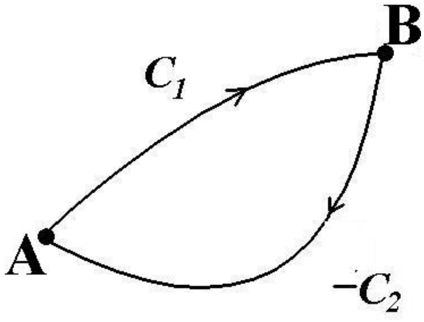
In general, suppose we have two different paths connecting point **A** to point **B**. Call these paths C_1 and C_2.



If we reverse direction along path C_2, we get a closed loop C.



$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{-C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} - \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

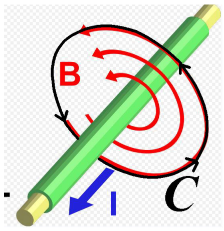


$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} + \int_{-C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} - \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 0 \text{ if and only if } \int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

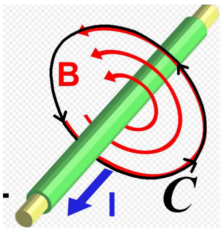
Other examples of closed loop integrals

Ampere's Law relates the current I along a wire to the magnetic field B around the wire.



If C is a closed loop around the wire, the circulation of the magnetic field B is proportional to the current I.

$$\oint_C \vec{\mathbf{B}} \bullet d\vec{\mathbf{r}} = \mu_0 I$$



If $\vec{\mathbf{v}}$ is the velocity vector field of a fluid, one of the basic assumptions is that the circulation is zero.

$$\oint_C \vec{\mathbf{v}} \bullet d\vec{\mathbf{r}} = 0$$