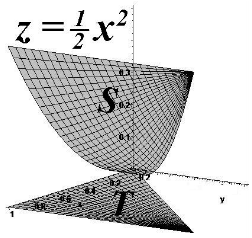
MA 441 Fall 2021 **Dr. E. Jacobs**

Practice Problems for Exam I - Solutions

---

1. Let $\mathcal{T}$ be the triangular region in the xy plane with vertices $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$ and $(1,\ 1,\ 0)$. Let S be the portion of the surface $z = \frac{1}{2}x^2$ that lies directly above $\mathcal{D}$



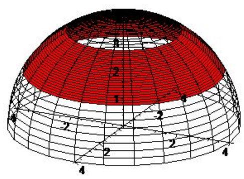
Find the surface area of S. Show all work.

$$A = \int_{\mathcal{T}} \sqrt{1 + z_x^2 + z_y^2}\, dy\, dx = \int_0^1 \int_0^x \left(1 + x^2\right)^{1/2}\, dy\, dx = \frac{1}{3}(2\sqrt{2} - 1)$$

---

2. Let S be the portion of a sphere described by the following equation:

$$\langle x,\ y,\ z\rangle = \langle 4\cos\theta\sin\phi,\ 4\sin\theta\sin\phi,\ 4\cos\phi\rangle \quad \text{for } 0 \le \theta \le 2\pi \text{ and } \frac{\pi}{6} \le \phi \le \frac{\pi}{3}$$



a. Find the surface area of S

$$dS = |\vec{\mathbf{r}}_\phi \times \vec{\mathbf{r}}_\theta|\, d\theta\, d\phi = 16\sin\phi\, d\theta\, d\phi$$

$$A(S) = \int_{\pi/6}^{\pi/3} \int_0^{2\pi} 16\sin\phi\, d\theta\, d\phi = 16\pi(\sqrt{3} - 1)$$

b. Suppose mass is distributed along S with a density function given by $\delta = \frac{1}{8}z$ kg/m^2. Find the total mass on S.

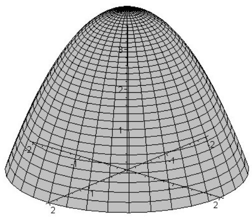
$$M = \iint_S \delta\, dS = \int_{\pi/6}^{\pi/3} \int_0^{2\pi} \frac{1}{8} \cdot 4\cos\phi \cdot 16\sin\phi\, d\theta\, d\phi = 4\pi$$

---

3. Let Ω be the portion of the parabolic surface $z = 1 - x^2 - y^2$ for $z \geq 0$. It can also be described by the parametric equation:

$$\vec{\mathbf{r}} = \left\langle u\cos\theta,\ u\sin\theta,\ 1 - u^2 \right\rangle \quad \text{for } 0 \leq u \leq 1 \text{ and } 0 \leq \theta \leq 2\pi$$

Let B be the disk of radius 1 in the xy plane centered around the origin. Let S be the closed surface consisting of Ω and B.



Let $\vec{\mathbf{F}} = \langle -y,\ x,\ z+1 \rangle$.

Calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ by adding $\iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ to $\iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$.

On B, $\vec{\mathbf{F}} = \langle -y,\ x,\ 1 \rangle$ and $\vec{\mathbf{n}}\, dS = \langle 0,\ 0,\ -1 \rangle\, dS$ so $\vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = -1\, dS$

$$\iint_B \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = (-1) \iint_B dS = -\text{Area}(B) = -\pi$$

On Ω, $\vec{\mathbf{n}}\, dS = \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\, d\theta\, du = \left\langle 2u^2\cos\theta,\ 2u^2\cos\theta,\ 2u^2\sin\theta,\ u \right\rangle\, d\theta\, du$

$$\begin{aligned} \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \left\langle -u\sin\theta,\ u\cos\theta,\ 2 - u^2 \right\rangle \bullet \left\langle 2u^2\cos\theta,\ 2u^2\cos\theta,\ 2u^2\sin\theta,\ u \right\rangle\, d\theta\, du \\ &= \left(2u - u^3\right)\, d\theta\, du \end{aligned}$$

$$\iint_\Omega \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \int_0^1 \int_0^{2\pi} \left(2u - u^3\right)\, d\theta\, du = \frac{3\pi}{2}$$

Therefore,

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = -\pi + \frac{3\pi}{2} = \frac{\pi}{2}$$

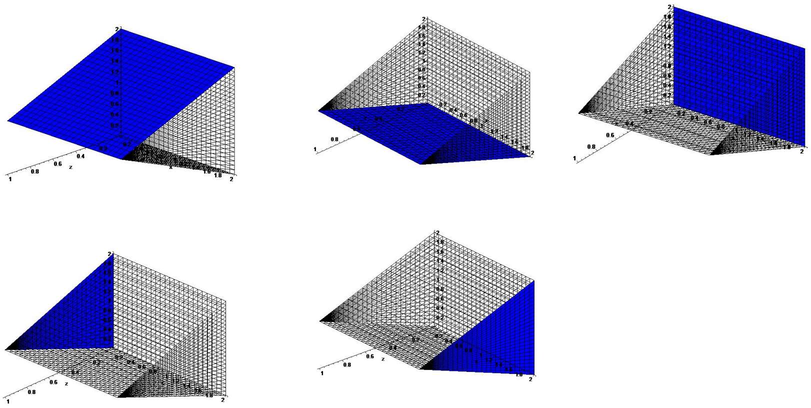
---

4. Let $\vec{\mathbf{F}}$ and surface S be defined exactly as in problem 3. Use the Divergence Theorem to calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iiint_V \nabla \bullet \vec{\mathbf{F}}\, dV = \int_0^1 \int_0^{2\pi} \int_0^{1-u^2} 1\, u\, dz\, d\theta\, du = \frac{\pi}{2}$$

---

5. Let V be the three dimensional region below the plane $z = 2 - x$ and above the plane $z = x$ for $0 \le x \le 1$ and $0 \le y \le 2$. Let S be the *closed surface* surrounding V. There are five surfaces that make up S.



Let $\vec{\mathbf{F}} = \langle x,\ 0,\ z\rangle$. Calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ by adding up the surface integrals for all five surfaces.

Hint: You will save some time if you can see which of these five surface integrals will equal 0.

Let T be the top surface (the part along $z = 2 - x$). The surface integrals over the other four boundaries are zero, so we only have to integrate over surface T.

$$\iint_T \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \int_0^1 \int_0^2 \langle x,\ 0,\ 2-x\rangle \bullet \langle 1,\ 0,\ 1\rangle\, dy\, dx = \int_0^1 \int_0^2 2\, dy\, dx = 4$$

---

6. Let $\vec{\mathbf{F}}$ and S be defined exactly as in problem 5. Calculate $\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS$ using the Divergence Theorem.

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iiint_V \nabla \bullet \vec{\mathbf{F}}\, dz\, dy\, dx = \int_0^1 \int_0^2 \int_x^{2-x} 2\, dz\, dy\, dx = 4$$

---

7. Let $\phi(x, y) = xy - \frac{x^2}{2}$ and $\vec{\mathbf{F}} = \left(x - y^2\right)\vec{\mathbf{j}} + \left(z - x^2\right)\vec{\mathbf{k}}$.

a)

$$\operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \left\langle 0,\ x - y^2,\ z - x^2 \right\rangle = -2y + 1$$

b)

$$\operatorname{curl} \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x - y^2 & z - x^2 \end{vmatrix} = (0)\,\vec{\mathbf{i}} - (-2x)\,\vec{\mathbf{j}} + (1)\,\vec{\mathbf{k}} = 2x\vec{\mathbf{j}} + \vec{\mathbf{k}}$$

c)

$$\nabla\phi = \frac{\partial}{\partial x}\left(xy - \frac{x^2}{2}\right)\vec{\mathbf{i}} + \frac{\partial}{\partial y}\left(xy - \frac{x^2}{2}\right)\vec{\mathbf{j}} = (y - x)\vec{\mathbf{i}} + x\vec{\mathbf{j}}$$

d)

$$\nabla^2\phi = \frac{\partial^2}{\partial x^2}\left(xy - \frac{x^2}{2}\right) + \frac{\partial^2}{\partial y^2}\left(xy - \frac{x^2}{2}\right) = -1$$