Surface Integral Examples

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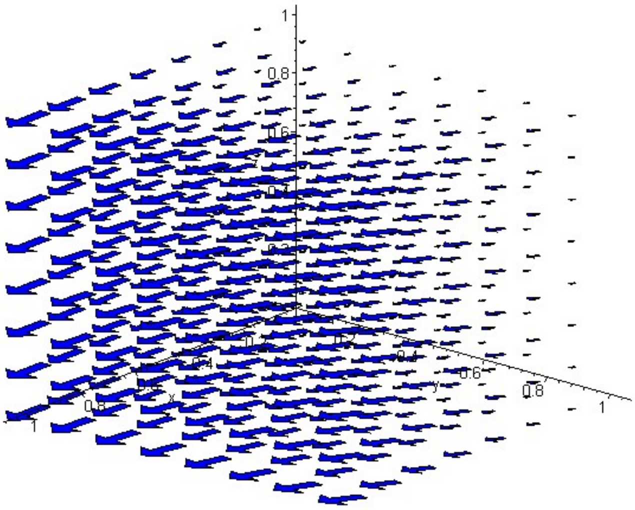
$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{\mathcal{D}} \vec{\mathbf{F}} \bullet \left(\frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right) du\, dv$$

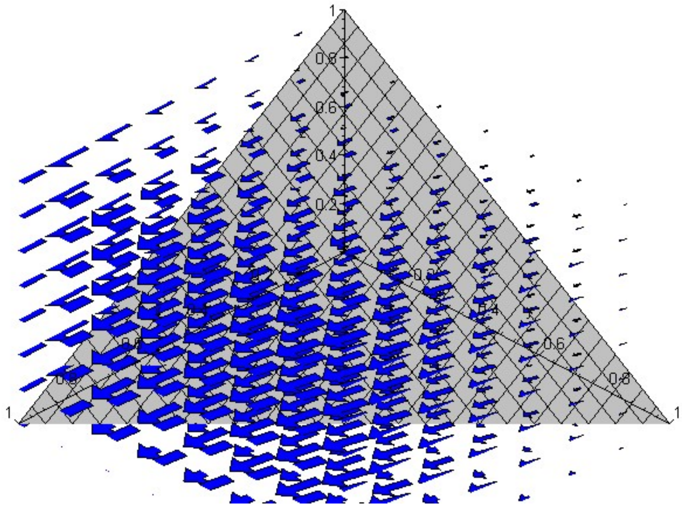
Example:

$$\vec{\mathbf{F}} = \langle 6x - xy,\ xy,\ 0 \rangle$$



$$\vec{\mathbf{F}} = \langle 6x - xy,\ xy,\ 0\rangle$$

Surface: $z = 1 - x - y$ in the first octant



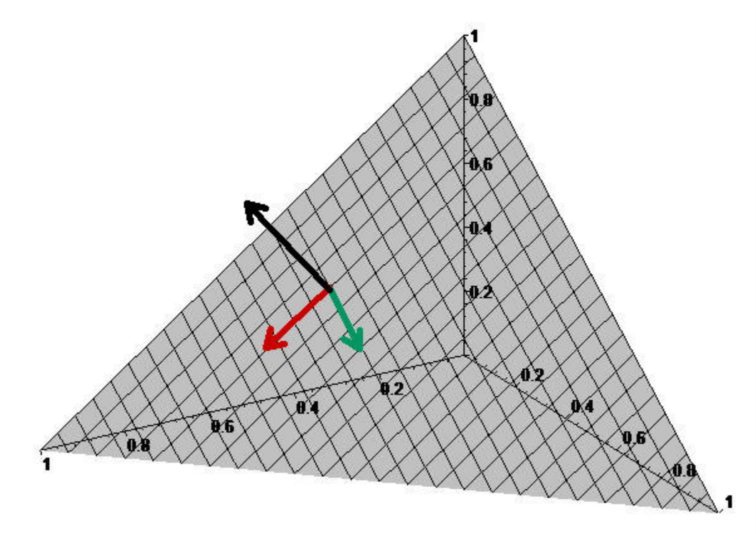
$$\text{Surface:} \quad z = 1 - x - y \quad \text{in the first octant}$$

$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle x,\ y,\ 1 - x - y \rangle$$

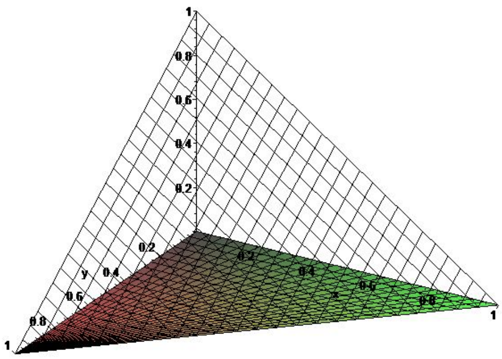
$$\frac{\partial \vec{\mathbf{r}}}{\partial x} = \langle 1,\ 0,\ -1 \rangle$$

$$\frac{\partial \vec{\mathbf{r}}}{\partial y} = \langle 0,\ 1,\ -1 \rangle$$

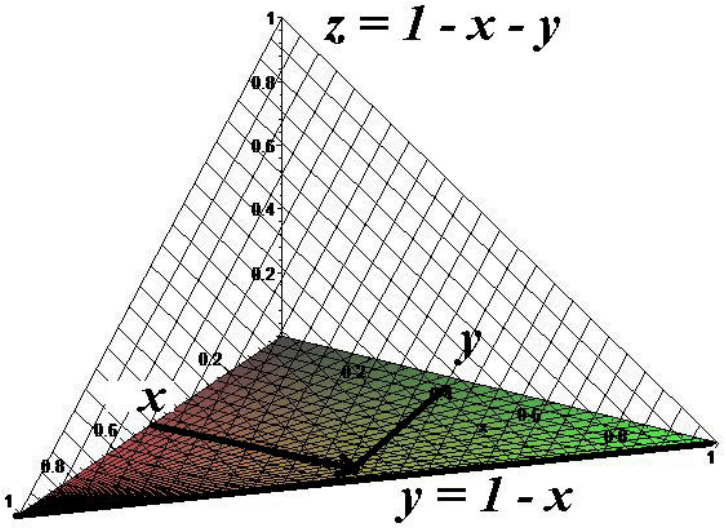
$$\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1,\ 1,\ 1 \rangle$$



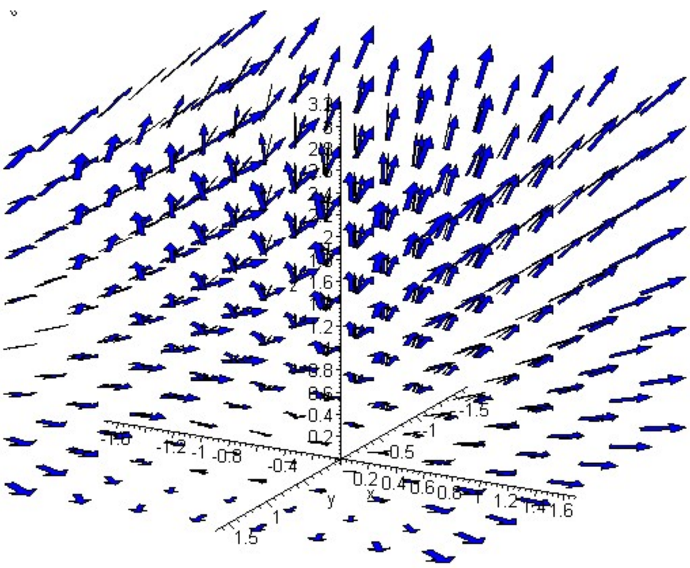
$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{\mathcal{D}} \langle 6x - xy,\ xy,\ 0\rangle \bullet \langle 1,\ 1,\ 1\rangle\, dy\, dx$$
$$= \iint_{\mathcal{D}} 6x\, dy\, dx$$



$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{\mathcal{D}} 6x\, dy\, dx = \int_0^1 \int_0^{1-x} 6x\, dy\, dx = 2$$

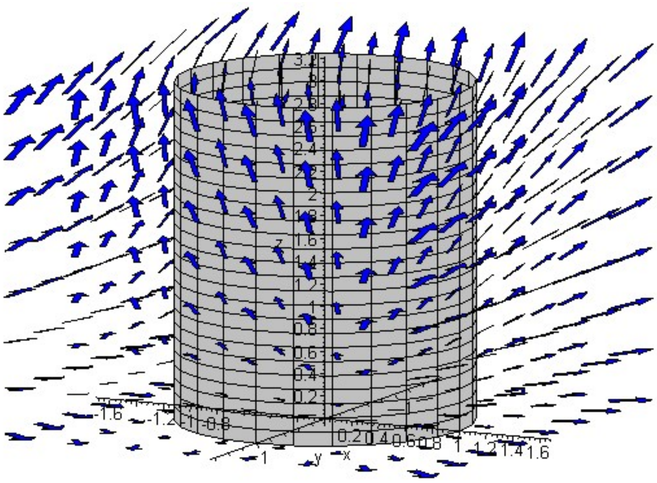


$$\vec{\mathbf{F}} = \langle x,\ y^2,\ z \rangle$$



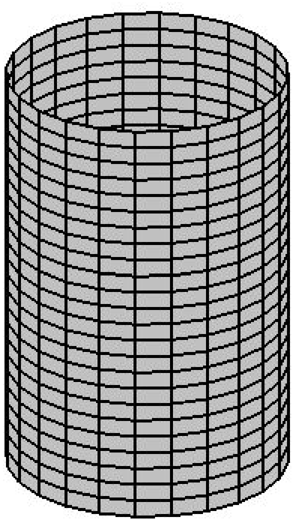
$$\vec{\mathbf{F}} = \langle x,\ y^2,\ z\rangle$$

Surface: $x^2 + y^2 = 1$ for $0 \le z \le 3$

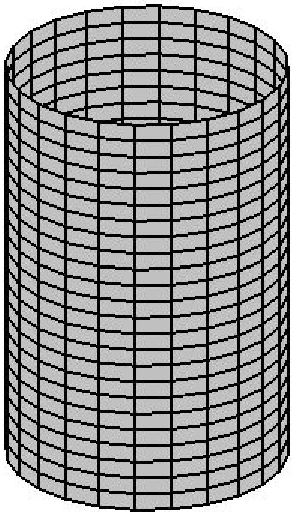


Surface: $x^2 + y^2 = 1$ for $0 \leq z \leq 3$

$$x = r\cos\theta \qquad y = r\sin\theta$$



$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle \cos\theta,\ \sin\theta,\ z \rangle$$

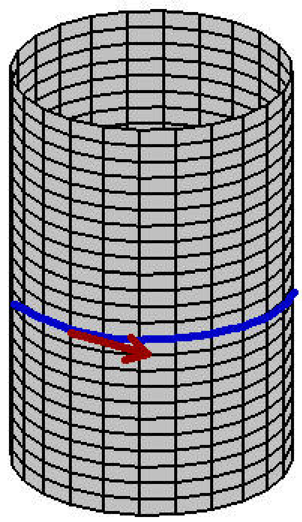


$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle \cos\theta,\ \sin\theta,\ z \rangle$$

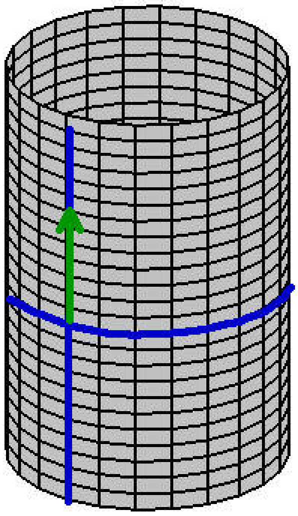
$$\frac{\partial \vec{\mathbf{r}}}{\partial \theta} = \langle -\sin\theta,\ \cos\theta,\ 0 \rangle$$

$$\frac{\partial \vec{\mathbf{r}}}{\partial z} = \langle 0,\ 0,\ 1 \rangle$$

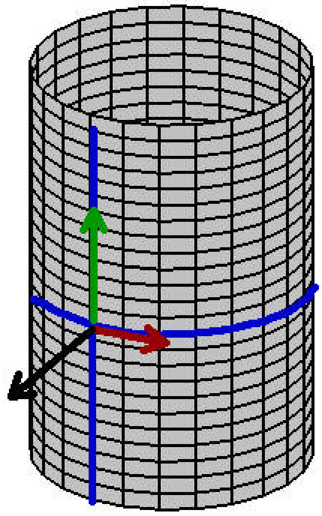
Hold z fixed and vary θ



Hold θ fixed and vary z



$$\frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial z} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta,\ \sin\theta,\ 0 \rangle$$



$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle \cos\theta,\ \sin\theta,\ z\rangle$$

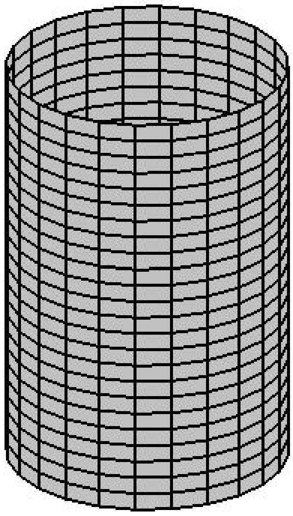
$$x = \cos\theta \qquad y = \sin\theta$$

$$\vec{\mathbf{F}} = \langle x,\ y^2,\ z\rangle = \langle \cos\theta,\ \sin^2\theta,\ z\rangle$$

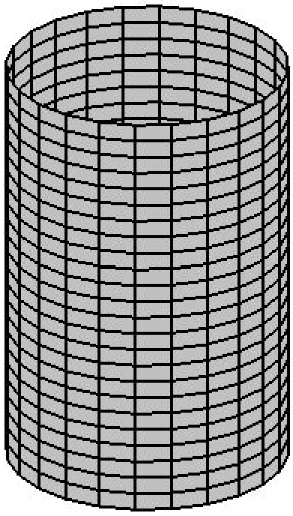
$$\begin{aligned}\vec{\mathbf{F}} \bullet \left(\frac{\partial\vec{\mathbf{r}}}{\partial\theta} \times \frac{\partial\vec{\mathbf{r}}}{\partial z}\right) &= \langle \cos\theta,\ \sin^2\theta,\ z\rangle \bullet \langle \cos\theta,\ \sin\theta,\ 0\rangle \\ &= \cos^2\theta + \sin^3\theta\end{aligned}$$

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{\mathcal{D}} \left(\cos^2\theta + \sin^3\theta\right)\, dz\, d\theta$$

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{\mathcal{D}} \left(\cos^2\theta + \sin^3\theta\right)\, dz\, d\theta$$

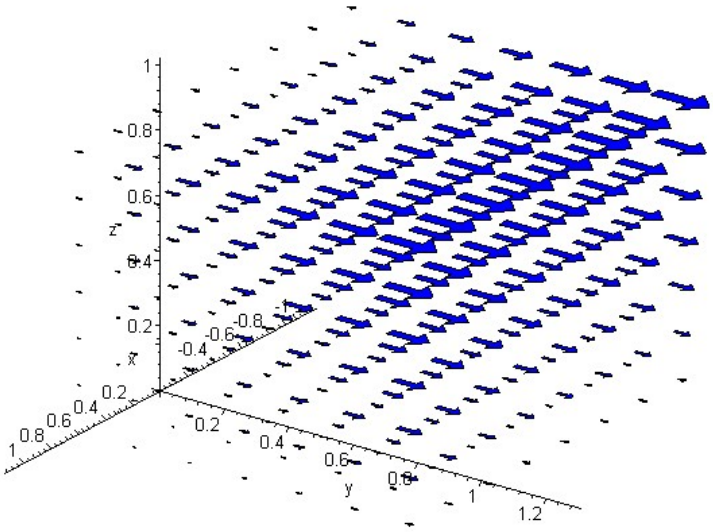


$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \int_0^{2\pi} \int_0^3 \left(\cos^2\theta + \sin^3\theta\right)\, dz\, d\theta$$



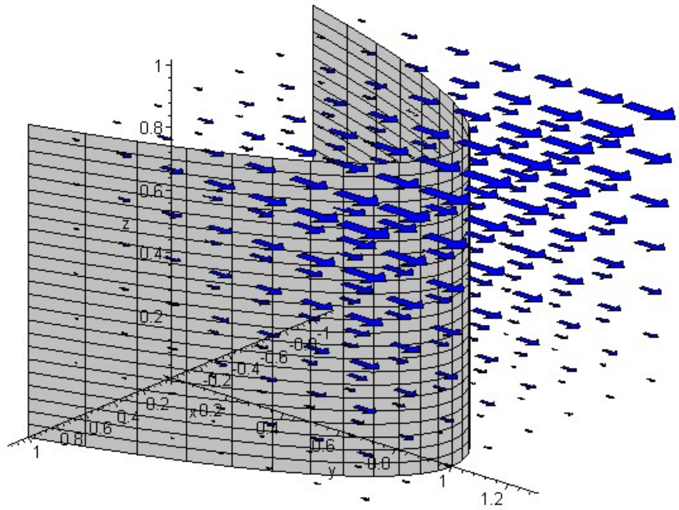
$$
\begin{aligned}
\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \int_0^{2\pi}\int_0^3 \left(\cos^2\theta + \sin^3\theta\right)\, dz\, d\theta \\
&= 3\int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2} + (1-\cos^2\theta)\sin\theta\right)\, d\theta \\
&= 3\pi
\end{aligned}
$$

$$\vec{\mathbf{F}} = \langle 0,\ yz,\ 0 \rangle$$



$$\vec{\mathbf{F}} = \langle 0,\ yz,\ 0 \rangle$$

Surface: $y = 1 - x^2$ for $-1 \le x \le 1$ and $0 \le z \le 1$



If $y = 1 - x^2$ then:

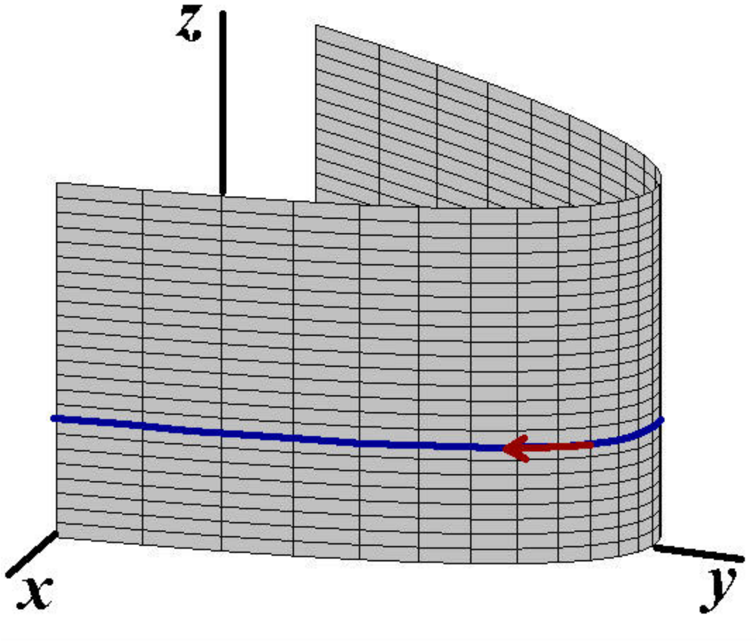
$$\vec{\mathbf{r}} = \langle x,\ y,\ z \rangle = \langle x,\ 1 - x^2,\ z \rangle$$

$$\frac{\partial \vec{\mathbf{r}}}{\partial x} = \langle 1,\ -2x,\ 0 \rangle$$

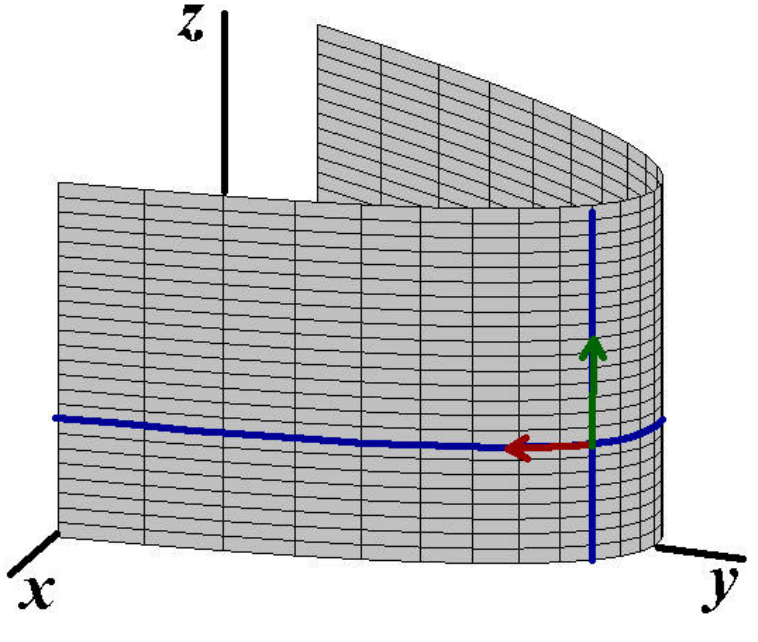
$$\frac{\partial \vec{\mathbf{r}}}{\partial z} = \langle 0,\ 0,\ 1 \rangle$$

In what order do we take the cross product?

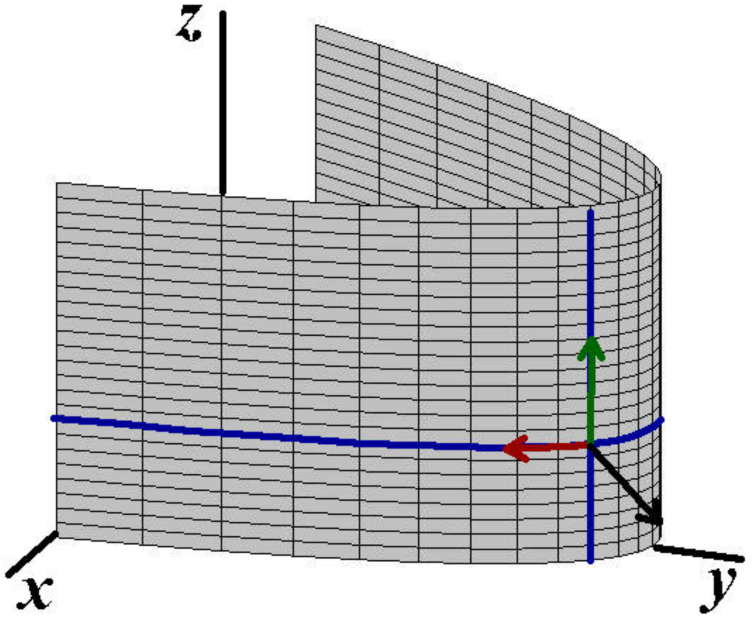
Hold z constant and increase x



Hold x constant and increase z



Use the right hand rule to get the cross product



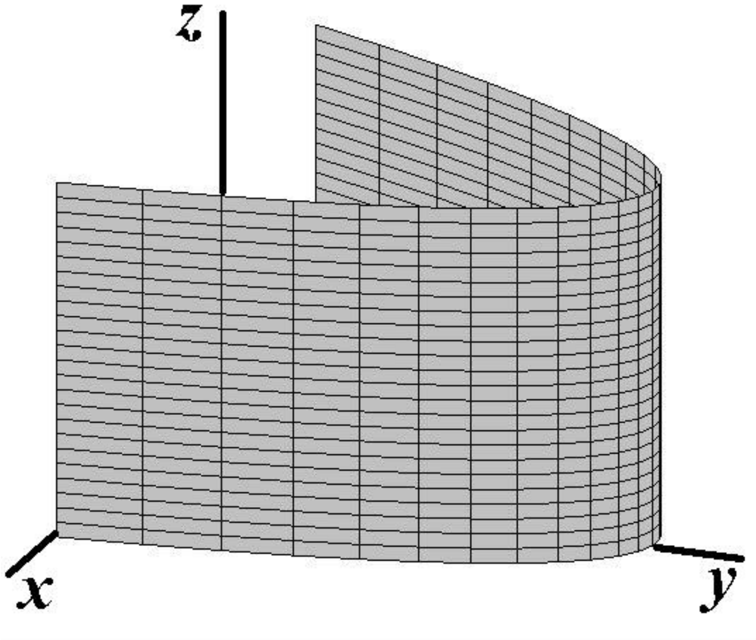
$$\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle = \langle x,\ 1-x^2,\ z\rangle$$

$$\frac{\partial \vec{\mathbf{r}}}{\partial z} \times \frac{\partial \vec{\mathbf{r}}}{\partial x} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 0 & 0 & 1 \\ 1 & -2x & 0 \end{vmatrix} = \langle 2x,\ 1,\ 0\rangle$$

$$\vec{\mathbf{F}} = \langle 0,\ yz,\ 0\rangle = \langle 0, (1-x^2)z,\ 0\rangle$$

$$\vec{\mathbf{F}} \bullet \left(\frac{\partial \vec{\mathbf{r}}}{\partial z} \times \frac{\partial \vec{\mathbf{r}}}{\partial x} \right) = \langle 0, (1 - x^2)z,\ 0 \rangle \bullet \langle 2x,\ 1,\ 0 \rangle = \left(1 - x^2\right) z$$

$$\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS = \iint_{\mathcal{D}} \left(1 - x^2\right) z\, dz\, dx$$



$$
\begin{aligned}
\iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{n}}\, dS &= \int_{-1}^{1}\int_{0}^{1} \left(1 - x^2\right) z\, dz\, dx \\
&= \int_{-1}^{1} \left[\left(1 - x^2\right) \frac{z^2}{2}\right]_{z=0}^{1} dx \\
&= \frac{1}{2}\int_{-1}^{1} \left(1 - x^2\right)\, dx \\
&= \frac{2}{3}
\end{aligned}
$$