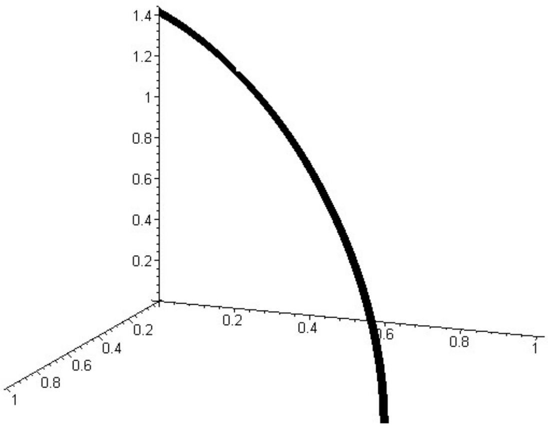
Review of Line Integrals

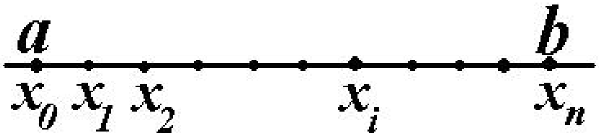
Dr. Elliott Jacobs

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



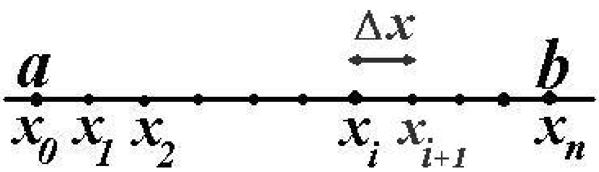
(force)(distance)

Divide the interval from a to b into n subintervals



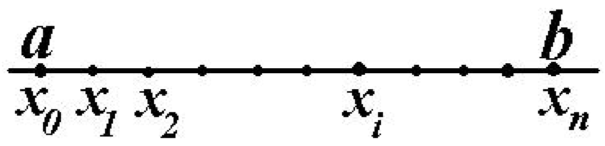
The work done in moving an object from x_i to x_{i+1} is approximately:

$$F(x_i)\,\Delta x$$



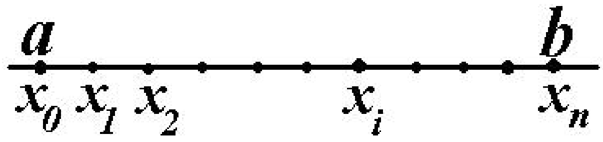
The total amount of work done in from a to b is approximately:

$$\sum_{i=0}^{n-1} F(x_i)\,\Delta x$$



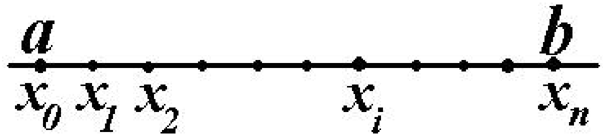
The exact total amount of work done is:

$$W = \lim_{n \to \infty} \sum_{i=0}^{n-1} F(x_i)\, \Delta x$$



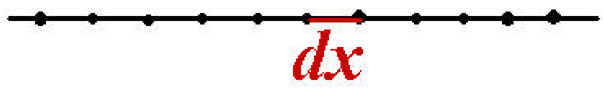
The exact total amount of work done is:

$$W = \lim_{n \to \infty} \sum_{i=0}^{n-1} F(x_i)\,\Delta x = \int_a^b F(x)\,dx$$



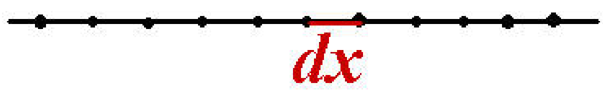
Work done along one small segment

$$F(x)\,dx$$

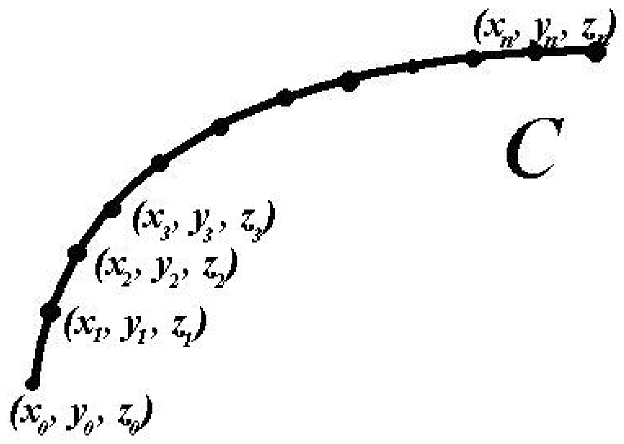


Total amount of work:

$$W = \int_a^b F(x)\, dx$$

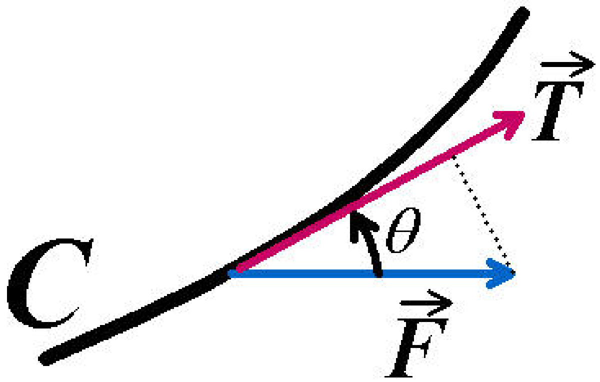


The path could be in higher dimension



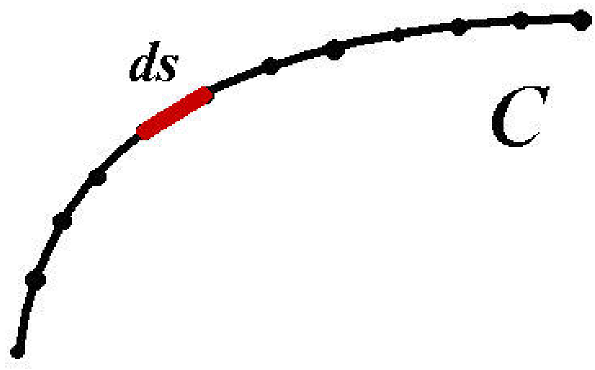
The calculation of work should use only the tangential component of force

$$\vec{\mathbf{F}} \bullet \vec{\mathbf{T}}$$



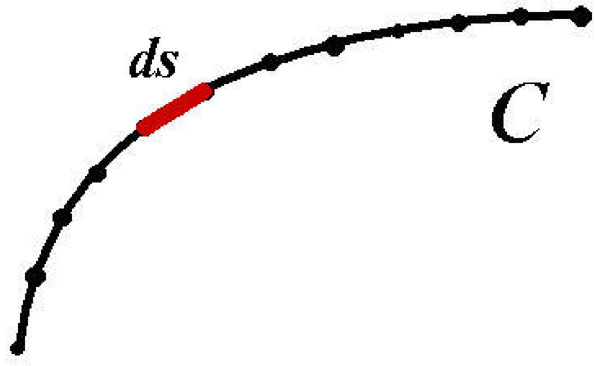
Work done along one small segment of length ds

$$\vec{\mathbf{F}} \bullet \vec{\mathbf{T}}\, ds$$



Total work along curve C

$$W = \int_C \vec{\mathbf{F}} \bullet \vec{\mathbf{T}}\, ds$$



Recall:

$$\frac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{v}} \qquad\qquad \frac{ds}{dt} = |\vec{\mathbf{v}}|$$

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$$d\vec{\mathbf{r}} = \vec{\mathbf{v}}\,dt \qquad ds = |\vec{\mathbf{v}}|\,dt$$

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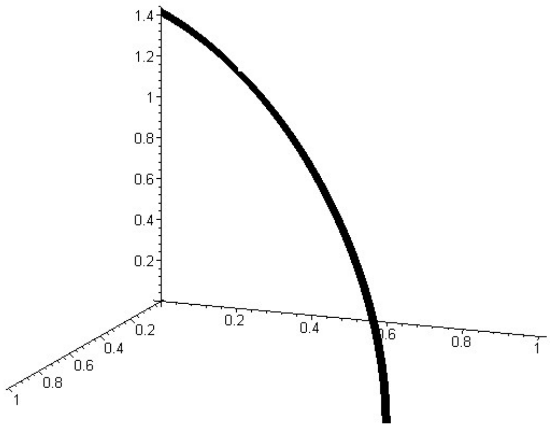
$$\begin{aligned}
\vec{\mathbf{F}} \bullet \vec{\mathbf{T}}\,ds &= \vec{\mathbf{F}} \bullet \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} \cdot |\vec{\mathbf{v}}|\,dt \\
&= \vec{\mathbf{F}} \bullet \vec{\mathbf{v}}\,dt \\
&= \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}
\end{aligned}$$

$$W = \int_C \vec{\mathbf{F}} \bullet \vec{\mathbf{T}}\, ds = \int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

Example:

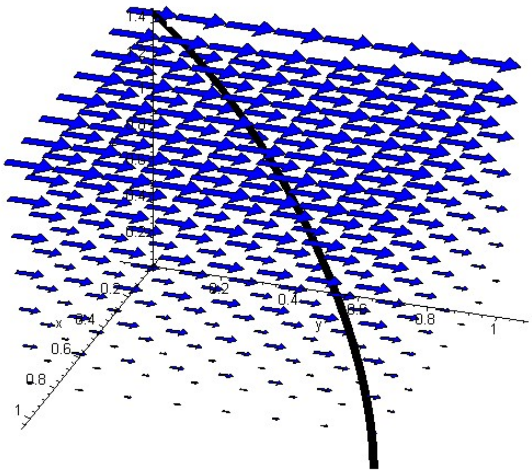
Let $\vec{\mathbf{r}} = \langle x(t),\ y(t),\ z(t)\rangle = \langle \sin t,\ \sin t,\ \cos t\rangle$

where $0 \leq t \leq \frac{\pi}{2}$



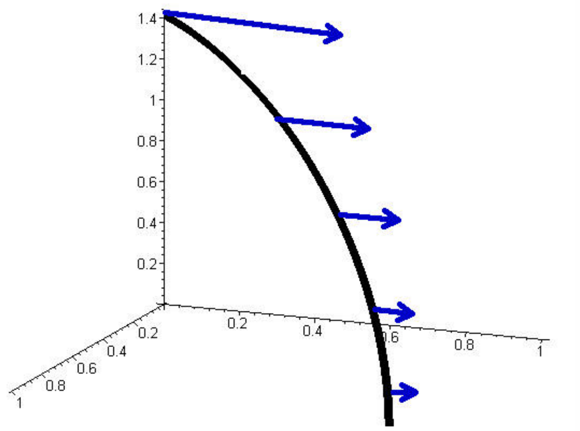
Example:

$$\vec{\mathbf{r}} = \langle \sin t,\ \sin t,\ \cos t \rangle \qquad \vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle = z\vec{\mathbf{j}}$$



Example:

$$\vec{\mathbf{r}} = \langle \sin t,\ \sin t,\ \cos t \rangle \qquad \vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle = z\vec{\mathbf{j}}$$



$$\vec{\mathbf{r}} = \langle \sin t,\ \sin t,\ \cos t \rangle \qquad \vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle = z\vec{\mathbf{j}}$$

$$d\vec{\mathbf{r}} = \frac{d\vec{\mathbf{r}}}{dt}\, dt = \langle \cos t,\ \cos t,\ -\sin t \rangle\, dt$$

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^{\pi/2} \langle 0,\ \cos t,\ 0 \rangle \bullet \langle \cos t,\ \cos t,\ -\sin t \rangle\, dt$$

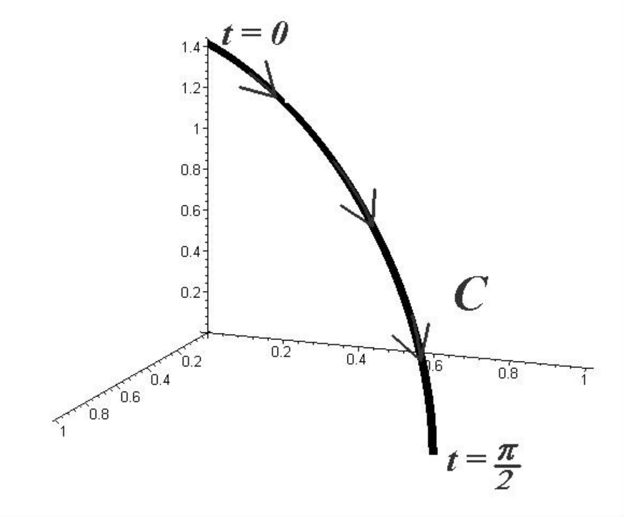
$$= \int_0^{\pi/2} \cos^2 t\, dt = \frac{\pi}{4}$$

What does it mean to reverse the limits of integration?

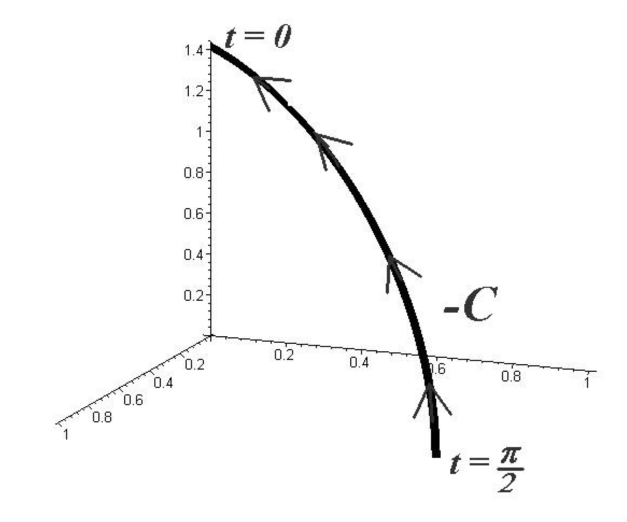
$$\int_0^{\pi/2} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = \frac{\pi}{4}$$

$$\int_{\pi/2}^{0} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = -\frac{\pi}{4}$$

$$\int_0^{\pi/2} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = \frac{\pi}{4}$$



$$\int_{\pi/2}^{0} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = -\frac{\pi}{4}$$



Reversing the limits of integration changes the direction that the path is traversed.

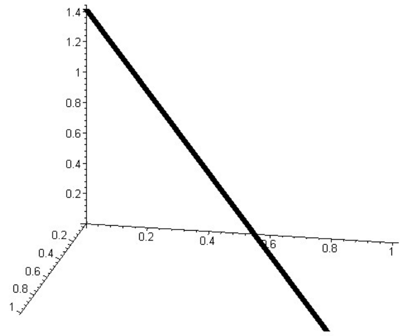
$$\int_{\pi/2}^{0} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = -\frac{\pi}{4} \qquad \int_{0}^{\pi/2} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = \frac{\pi}{4}$$

$$\int_{-C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = -\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

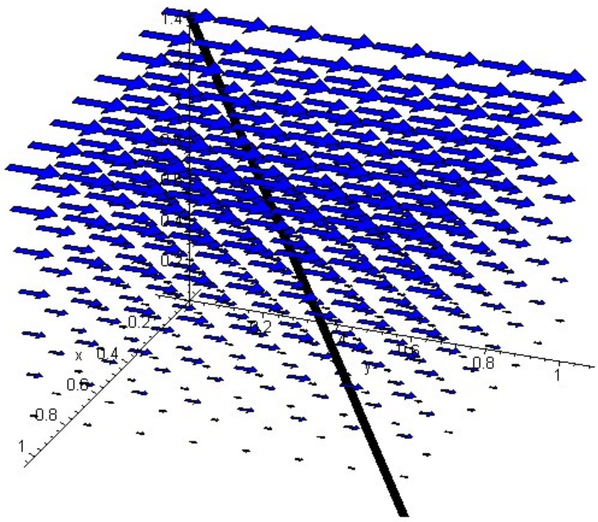
What happens if we change the path between (0, 0, 1) and (1, 1, 0)?

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$$

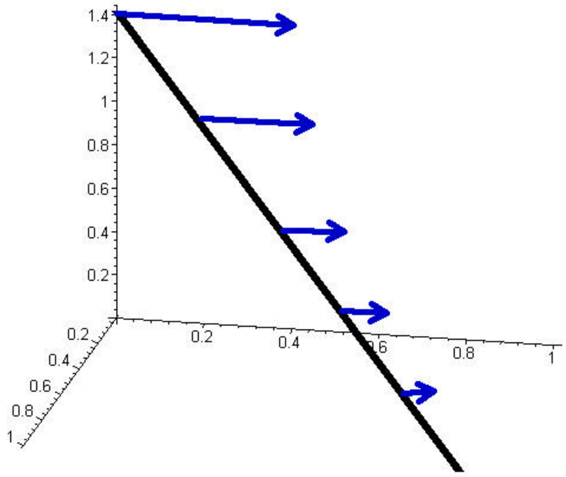
$\vec{\mathbf{r}} = \langle 0,\ 0,\ 1\rangle + t\langle 1,\ 1,\ -1\rangle = \langle t,\ t,\ 1-t\rangle$
where $0 \le t \le 1$



$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle = z\vec{\mathbf{j}}$$



$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle$$



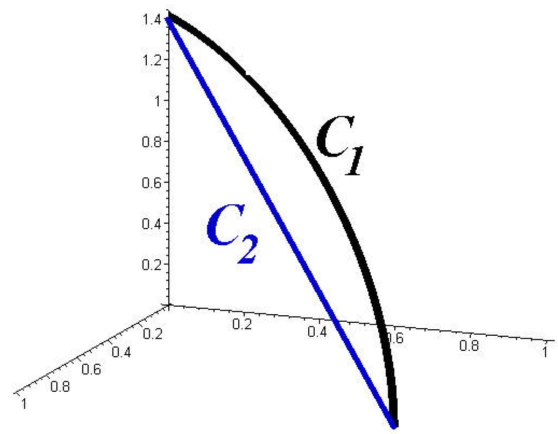
$$\vec{\mathbf{F}} = \langle 0,\ z,\ 0 \rangle = \langle 0,\ 1 - t,\ 0 \rangle$$

$$\vec{\mathbf{r}} = \langle t,\ t,\ 1 - t \rangle \qquad d\vec{\mathbf{r}} = \frac{d\vec{\mathbf{r}}}{dt}\, dt = \langle 1,\ 1,\ -1 \rangle\, dt$$

$$\int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^1 (1 - t)\, dt = \frac{1}{2}$$

For $\vec{\mathbf{F}} = z\vec{\mathbf{j}}$,

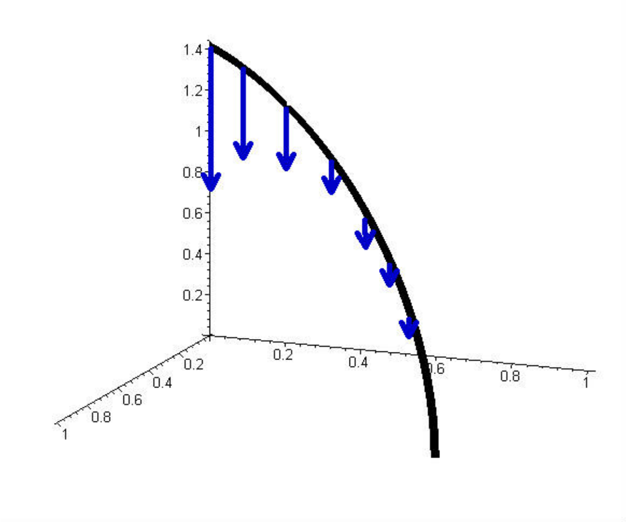
$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} \neq \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



Let's try this with a different vector field

$$\vec{\mathbf{F}} = -z\vec{\mathbf{k}} = \langle 0,\ 0,\ -z \rangle$$

$$\vec{\mathbf{r}} = \langle \sin t,\ \sin t,\ \cos t \rangle \qquad 0 \le t \le \frac{\pi}{2}$$



$$\vec{\mathbf{F}} = -z\vec{\mathbf{k}} = \langle 0,\ 0,\ -z \rangle = \langle 0,\ 0,\ -\cos t \rangle$$

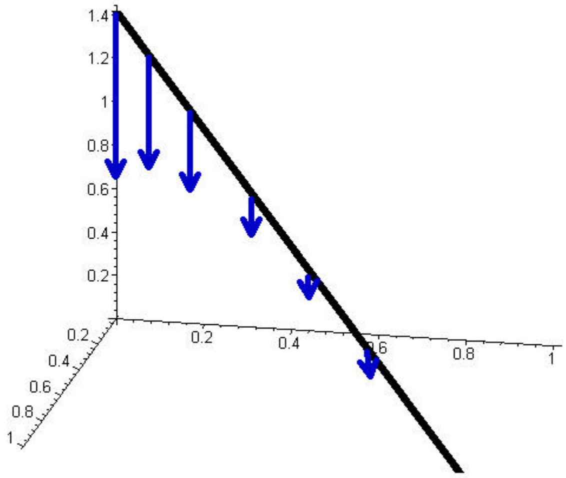
$$\vec{\mathbf{r}} = \langle \sin t,\ \sin t,\ \cos t \rangle \qquad 0 \le t \le \frac{\pi}{2}$$

$$d\vec{\mathbf{r}} = \frac{d\vec{\mathbf{r}}}{dt}\, dt = \langle \cos t,\ \cos t,\ -\sin t \rangle\, dt$$

$$\vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \sin t \cos t\, dt$$

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^{\pi/2} \sin t \cos t\, dt = \frac{1}{2}$$

Try $\vec{\mathbf{F}} = -z\vec{\mathbf{k}}$ along the straight line path C_2



$$\vec{\mathbf{r}} = \langle t,\ t,\ 1-t \rangle \qquad d\vec{\mathbf{r}} = \langle 1,\ 1,\ -1 \rangle\, dt$$

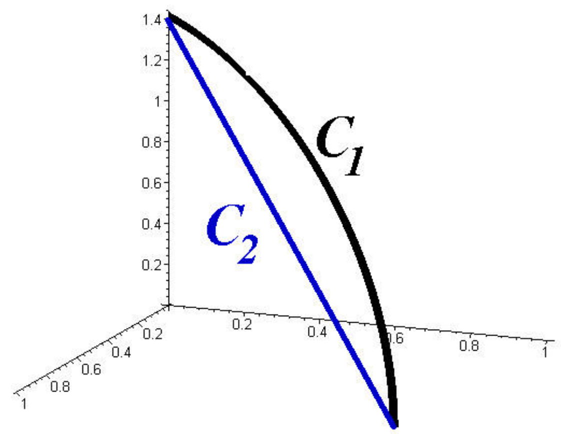
$$\vec{\mathbf{F}} = \langle 0,\ 0,\ -z \rangle = \langle 0,\ 0,\ -(1-t) \rangle$$

$$\vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = (1-t)\, dt$$

$$\int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^1 (1-t)\, dt = \frac{1}{2}$$

This time, for $\vec{\mathbf{F}} = -z\vec{\mathbf{k}}$, the path doesn't seem to matter.

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$



Given a vector field $\vec{\mathbf{F}}$ and two different paths, C_1 and C_2, connecting the same two points, how can we tell when the path doesn't matter?

$$\int_{C_1} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{C_2} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$